

# Twisted Dirac Operators and the Noncommutative Residue for Manifolds with Boundary

Jian Wang<sup>a,b</sup>, Yong Wang<sup>a,\*</sup>

<sup>a</sup>*School of Mathematics and Statistics, Northeast Normal University, Changchun, 130024, P.R.China*

<sup>b</sup>*Chengde Petroleum College, Chengde, 067000, P.R.China*

---

## Abstract

In this paper, we give two Lichnerowicz type formulas for Dirac operators and signature operators twisted by a vector bundle with a non-unitary connection. We also prove two Kastler-Kalau-Walze type theorems for twisted Dirac operators and twisted signature operators on 4-dimensional manifolds with (resp. without) boundary.

*Keywords:* twisted Dirac operators; twisted signature operators; noncommutative residue; non-unitary connection.

---

## 1. Introduction

The noncommutative residue found in [1, 2] plays a prominent role in noncommutative geometry. For one-dimensional manifolds, the noncommutative residue was discovered by Adler[3] in connection with geometric aspects of nonlinear partial differential equations. For arbitrary closed compact  $n$ -dimensional manifolds, the noncommutative residue was introduced by Wodzicki in [2] using the theory of zeta functions of elliptic pseudodifferential operators. In [4], Connes used the noncommutative residue to derive a conformal 4-dimensional Polyakov action analogy. In [5], Connes proved that the noncommutative residue on a compact manifold  $M$  coincided with Dixmier's trace on pseudodifferential operators of order  $-\dim M$ . Indeed, Alain Connes made a challenging observation that the Wodzicki residue of the inverse square of the (Atiyah-Singer-Lichnerowicz) Dirac operator yields the Einstein-Hilbert action of general relativity, which is called the Kastler-Kalau-Walze theorem now. Kastler[6] gave a brute-force proof of this theorem. Kalau and Walze[7] proved this theorem in the normal coordinates system simultaneously. Ackermann[8] gave a note on a new proof of this theorem by means of the heat kernel expansion.

Based on the theory of the noncommutative residue introduced by Wodzicki, Fedosov etc.[9] constructed a noncommutative residue on the algebra of classical elements in Boutet de Monvel's calculus on a compact manifold with boundary of dimension  $n > 2$ . For Dirac operators and signature operators on manifolds with boundary, Wang[10] gave an operator-theoretic explanation of the gravitational action for manifolds with boundary and proved a Kastler-Kalau-Walze type theorem. In [11], Wang computed the lower dimensional volume  $\text{Vol}^{(2,2)}$  for 5-dimensional and 6-dimensional spin manifolds with boundary and also got a Kastler-Kalau-Walze type theorem in this case. Then, we got a Kastler-Kalau-Walze type theorem associated with nonminimal operators by heat equation asymptotics on compact manifolds without boundary and proved Kastler-Kalau-Walze type theorems for foliations with or without boundary associated with sub-Dirac operators in [12] and [13] respectively.

On the other hand, Bismut and Zhang [14] introduced the de-Rham Hodge operator twisted by a flat vector bundle with a non-metric connection, and extended the famous Cheeger-Müller theorem to the

---

\*Corresponding author.

Email addresses: wangj484@nenu.edu.cn (Jian Wang), wangy581@nenu.edu.cn (Yong Wang)

non-unitary case. In [15], Ma and Zhang extended the Atiyah-Patodi-Singer  $\eta$ -Invariant to the twisted non-unitary flat vector bundle case. In [16], Zhang considered the sub-signature operators twisted by a non-unitary flat vector bundle and proved the associated Riemann-Roch theorem. Motivated by [14], [15], [16] and [10], we shall prove two Lichnerowicz type formulas for  $\tilde{D}_F^* \tilde{D}_F$  and  $\hat{D}_F^* \hat{D}_F$  (for definitions, see Section 2 and Section 4), and prove two Kastler-Kalau-Walze type theorems for  $\tilde{D}_F^* \tilde{D}_F$  and  $\hat{D}_F^* \hat{D}_F$  on manifolds with (resp. without) boundary.

This paper is organized as follows: In Section 2, we give a Lichnerowicz type formula for  $\tilde{D}_F^* \tilde{D}_F$ . A Kastler-Kalau-Walze type theorem for  $\tilde{D}_F^* \tilde{D}_F$  is given in Section 3. In Section 4 and Section 5, we give a Lichnerowicz type formula and prove a Kastler-Kalau-Walze type theorem for  $\hat{D}_F^* \hat{D}_F$ .

## 2. A Lichnerowicz formula for Dirac operators twisted by a vector bundle with a non-unitary connection

In this section we consider a  $n$ -dimensional oriented Riemannian manifold  $(M, g^M)$  equipped with a fixed spin structure. We recall twisted Dirac operators. Let  $S(TM)$  be the spinors bundle and  $F$  be an additional smooth vector bundle equipped with a non-unitary connection  $\tilde{\nabla}^F$ . Let  $\tilde{\nabla}^{F,*}$  be the dual connection on  $F$ , and define

$$\nabla^F = \frac{\tilde{\nabla}^F + \tilde{\nabla}^{F,*}}{2}, \quad \Phi = \frac{\tilde{\nabla}^F - \tilde{\nabla}^{F,*}}{2}, \quad (2.1)$$

then  $\nabla^F$  is a metric connection and  $\Phi$  is an endomorphism of  $F$  with a 1-form coefficient. We consider the tensor product vector bundle  $S(TM) \otimes F$ , which becomes a Clifford module via the definition:

$$c(a) = c(a) \otimes \text{id}_F, \quad a \in TM, \quad (2.2)$$

and which we equip with the compound connection:

$$\tilde{\nabla}^{S(TM) \otimes F} = \nabla^{S(TM)} \otimes \text{id}_F + \text{id}_{S(TM)} \otimes \tilde{\nabla}^F. \quad (2.3)$$

The corresponding twisted Dirac operator  $\tilde{D}_F$  is locally specified as follows:

$$\tilde{D}_F = \sum_{i=1}^n c(e_i) \tilde{\nabla}_{e_i}^{S(TM) \otimes F}. \quad (2.4)$$

Let

$$\nabla^{S(TM) \otimes F} = \nabla^{S(TM)} \otimes \text{id}_F + \text{id}_{S(TM)} \otimes \nabla^F, \quad (2.5)$$

then the spinor connection  $\tilde{\nabla}$  induced by  $\nabla^{S(TM) \otimes F}$  is locally given by

$$\tilde{\nabla}^{S(TM) \otimes F} = \nabla^{S(TM)} \otimes \text{id}_F + \text{id}_{S(TM)} \otimes \nabla^F + \text{id}_{S(TM)} \otimes \Phi. \quad (2.6)$$

Let

$$D_F = \sum_{i=1}^n c(e_i) \nabla_{e_i}^{S(TM) \otimes F}, \quad (2.7)$$

then the twisted Dirac operators  $\tilde{D}_F, \tilde{D}_F^*$  associated to the connection  $\tilde{\nabla}$  as follows.

**Definition 2.1.** For sections  $\psi \otimes \chi \in S(TM) \otimes F$ ,

$$\tilde{D}_F(\psi \otimes \chi) = D_F(\psi \otimes \chi) + \sum_{i=1}^n c(e_i) \otimes \Phi(e_i)(\psi \otimes \chi), \quad (2.8)$$

$$\tilde{D}_F^*(\psi \otimes \chi) = D_F(\psi \otimes \chi) - \sum_{i=1}^n c(e_i) \otimes \Phi^*(e_i)(\psi \otimes \chi). \quad (2.9)$$

Here  $\Phi^*(e_i)$  denotes the adjoint of  $\Phi(e_i)$ .

We first establish the main theorem in this section. Let  $c(\Phi) = \sum_{i=1}^n c(e_i) \otimes \Phi(e_i)$ ,  $c(\Phi^*) = \sum_{i=1}^n c(e_i) \otimes \Phi^*(e_i)$ , one has the following Lichnerowicz formula.

**Theorem 2.2.** *The following identity holds:*

$$\begin{aligned} \tilde{D}_F^* \tilde{D}_F &= -\left[ g^{ij} (\bar{\nabla}_{\partial_i} \bar{\nabla}_{\partial_j} - \bar{\nabla}_{\nabla_{\partial_i}^L \partial_j}) \right] + \frac{1}{4} s + \frac{1}{2} \sum_{i \neq j} R^F(e_i, e_j) c(e_i) c(e_j) - c(\Phi^*) c(\Phi) \\ &\quad + \frac{1}{2} \sum_j \left( \nabla_{e_j}^F c(\Phi^*) \right) c(e_j) + \frac{1}{2} \sum_j c(e_j) \nabla_{e_j}^F c(\Phi) + \frac{1}{4} \sum_i \left[ c(\Phi^*) c(e_i) - c(e_i) c(\Phi) \right]^2, \end{aligned} \quad (2.10)$$

where  $s$  is the scalar curvature,  $R^F$  denotes the curvature-tensor of  $\nabla^F$  on  $F$ , and for  $X \in \Gamma(M, TM)$

$$\bar{\nabla}_X = \nabla_X^{S(TM) \otimes F} + \frac{1}{2} [c(\Phi^*) c(X) - c(X) c(\Phi)]. \quad (2.11)$$

In order to prove Theorem 2.2, we recall the basic notions of Laplace type operators [17]. Let  $V$  be a vector bundle on  $M$ . Any differential operator  $P$  of Laplace type has locally the form

$$P = -(g^{ij} \partial_i \partial_j + A^i \partial_i + B) \quad (2.12)$$

where  $\partial_i$  is a natural local frame on  $TM$ , and  $(g^{ij})_{1 \leq i, j \leq n}$  is the inverse matrix associated with the metric matrix  $(g_{ij})_{1 \leq i, j \leq n}$  on  $M$ , and  $A^i$  and  $B$  are smooth sections of  $\text{End}(V)$  on  $M$  (Endomorphism). If  $P$  is a Laplace type operator of the form (2.12), then there is a unique connection  $\nabla$  on  $V$  and a unique Endomorphism  $E$  such that

$$P = -\left[ g^{ij} (\nabla_{\partial_i} \nabla_{\partial_j} - \nabla_{\nabla_{\partial_i}^L \partial_j}) + E \right], \quad (2.13)$$

where  $\nabla^L$  denotes the Levi-Civita connection on  $M$ . Moreover (with local frames of  $T^*M$  and  $V$ ),  $\nabla_{\partial_i} = \partial_i + \omega_i$  and  $E$  are related to  $g^{ij}$ ,  $A^i$ , and  $B$  through

$$\omega_i = \frac{1}{2} g_{ij} (A^i \partial^j + g^{kl} \Gamma_{kl}^j \text{Id}), \quad (2.14)$$

$$E = B - g^{ij} (\partial_i (\omega_j) + \omega_i \omega_j - \omega_k \Gamma_{ij}^k), \quad (2.15)$$

where  $\Gamma_{kl}^j$  is the Christoffel coefficient of  $\nabla^L$ .

The next task then is to prove  $\tilde{D}_F^* \tilde{D}_F$  has the Laplace type form. The twisted Dirac operator  $D_F$  is locally given as follows.

**Lemma 2.3.** *Let  $\{e_i\} (1 \leq i, j \leq n)$  ( $\{\partial_i\}$ ) be the orthonormal frames (natural frames respectively) on  $TM$ ,*

$$D_F = \sum_{i,j} g^{ij} c(\partial_i) \nabla_{\partial_j}^{S(TM) \otimes F} = \sum_j^n c(e_j) \nabla_{e_j}^{S(TM) \otimes F}, \quad (2.16)$$

where  $\nabla_{\partial_j}^{S(TM) \otimes F} = \partial_j + \sigma_j^s + \sigma_j^F$  and  $\sigma_j^s = \frac{1}{4} \sum_{j,k} \langle \nabla_{\partial_i}^L e_j, e_k \rangle c(e_j) c(e_k)$ ,  $\sigma_j^F$  is the connection matrix of  $\nabla^F$ .

Let  $\partial^j = g^{ij} \partial_i$ ,  $\sigma^i = g^{ij} \sigma_j$ ,  $\Gamma^k = g^{ij} \Gamma_{ij}^k$ . From (6a) in [6], we have

$$\begin{aligned} D_F^2 &= -g^{ij} \partial_i \partial_j - 2\sigma_{S(TM) \otimes F}^j \partial_j + \Gamma^k \partial_k - g^{ij} \left[ \partial_i (\sigma_{S(TM) \otimes F}^j) + \sigma_{S(TM) \otimes F}^i \sigma_{S(TM) \otimes F}^j \right. \\ &\quad \left. - \Gamma_{ij}^k \sigma_{S(TM) \otimes F}^k \right] + \frac{1}{4} s + \frac{1}{2} \sum_{i \neq j} R^F(e_i, e_j) c(e_i) c(e_j). \end{aligned} \quad (2.17)$$

where  $s$  is the scalar curvature.

We will computer  $\tilde{D}_F^* \tilde{D}_F$ . We note that

$$\tilde{D}_F^* \tilde{D}_F = D_F^2 - c(\Phi^*) D_F + D_F c(\Phi) - c(\Phi^*) c(\Phi), \quad (2.18)$$

and

$$\begin{aligned} -c(\Phi^*) D_F + D_F c(\Phi) &= -\sum_j c(\Phi^*) c(e_j) \left[ e_j + \sigma_j^{S(TM) \otimes F} \right] + \sum_j c(e_j) \otimes c(\Phi) e_j + \sum_j c(e_j) \otimes e_j (c(\Phi)) \\ &\quad + \sum_j \left[ c(e_j) \sigma_j^{S(TM)} \otimes c(\Phi) + c(e_j) \otimes \sigma_j^F c(\Phi) \right]. \end{aligned} \quad (2.19)$$

Combining (2.17)-(2.19), we obtain the specification of  $\tilde{D}_F^* \tilde{D}_F$ .

$$\begin{aligned} \tilde{D}_F^* \tilde{D}_F &= -g^{ij} \partial_i \partial_j - 2\sigma_{S(TM) \otimes F}^j \partial_j + \Gamma^k \partial_k - \sum_j \left[ c(\Phi^*) c(e_j) - c(e_j) \otimes c(\Phi) \right] e_j \\ &\quad - g^{ij} \left[ \partial_i (\sigma_{S(TM) \otimes F}^j) + \sigma_{S(TM) \otimes F}^i \sigma_{S(TM) \otimes F}^j - \Gamma_{ij}^k \sigma_{S(TM) \otimes F}^k \right] \\ &\quad - \sum_j \left[ c(\Phi^*) c(e_j) \right] \sigma_j^{S(TM) \otimes F} + \sum_j c(e_j) \otimes e_j (c(\Phi)) \\ &\quad + \sum_j \left[ c(e_j) \sigma_j^{S(TM)} \otimes c(\Phi) + c(e_j) \otimes \sigma_j^F c(\Phi) \right] - c(\Phi^*) c(\Phi) \\ &\quad + \frac{1}{4} s + \frac{1}{2} \sum_{i \neq j} R^F(e_i, e_j) c(e_i) c(e_j). \end{aligned} \quad (2.20)$$

In terms of local coordinates  $\{\partial_i\}$  inducing the coordinate transformation  $e_j = \sum_{k=1}^n \langle e_j, dx^k \rangle \partial_k$ , let  $\Gamma^k = g^{ij} \Gamma_{ij}^k$ , then

$$\omega_j = \sigma_{S(TM) \otimes F}^j + \frac{1}{2} \left( \sum_{k=1}^n \langle e_k, dx^j \rangle c(\Phi^*) c(e_k) - \sum_{k=1}^n \langle e_k, dx^j \rangle c(e_k) c(\Phi) \right) + \frac{1}{2} \Gamma^i. \quad (2.21)$$

For a smooth vector field  $X \in \Gamma(M, TM)$ , let  $c(X)$  denote the Clifford action. By direct computation in normal coordinates, we obtain

$$\bar{\nabla}_X = \nabla_X^{S(TM) \otimes F} + \frac{1}{2} [c(\Phi^*) c(X) - c(X) c(\Phi)]. \quad (2.22)$$

We now compute  $E$ . Regrouping the terms and inserting (2.21) into (2.15), we obtain

$$\begin{aligned}
E &= g^{ij} \left[ \partial_i (\sigma_{S(TM) \otimes F}^j) + \sigma_{S(TM) \otimes F}^i \sigma_{S(TM) \otimes F}^j - \Gamma_{ij}^k \sigma_{S(TM) \otimes F}^k \right] + \sum_j \left[ c(\Phi^*) c(e_j) \right] \sigma_j^{S(TM) \otimes F} \\
&\quad - \sum_j c(e_j) \otimes e_j (c(\Phi)) - \sum_j \left[ c(e_j) \sigma_j^{S(TM)} \otimes c(\Phi) + c(e_j) \otimes \sigma_j^F c(\Phi) \right] \\
&\quad + c(\Phi^*) c(\Phi) - \frac{1}{4} s - \frac{1}{2} \sum_{i \neq j} R^F(e_i, e_j) c(e_i) c(e_j) \\
&\quad - \partial^j (\sigma_{S(TM) \otimes F}^j) - \frac{1}{2} \partial^j \left( \sum_{k=1}^n \langle e_k, \mathbf{d}x^j \rangle c(\Phi^*) c(e_k) - \sum_{k=1}^n \langle e_k, \mathbf{d}x^j \rangle c(e_k) c(\Phi) \right) \\
&\quad - g^{ij} \sigma_{S(TM) \otimes F}^i \sigma_{S(TM) \otimes F}^j - \frac{1}{2} g^{ij} \sigma_{S(TM) \otimes F}^i \left( \sum_{k=1}^n \langle e_k, \mathbf{d}x^j \rangle c(\Phi^*) c(e_k) - \sum_{k=1}^n \langle e_k, \mathbf{d}x^j \rangle c(e_k) c(\Phi) \right) \\
&\quad - \frac{1}{2} g^{ij} \left( \sum_{k=1}^n \langle e_k, \mathbf{d}x^i \rangle c(\Phi^*) c(e_k) - \sum_{k=1}^n \langle e_k, \mathbf{d}x^i \rangle c(e_k) c(\Phi) \right) \sigma_j^{S(TM) \otimes F} \\
&\quad - \frac{1}{4} g^{ij} \left( \sum_{k=1}^n \langle e_k, \mathbf{d}x^i \rangle c(\Phi^*) c(e_k) - \sum_{k=1}^n \langle e_k, \mathbf{d}x^i \rangle c(e_k) c(\Phi) \right) \\
&\quad \times \left( \sum_{k=1}^n \langle e_k, \mathbf{d}x^j \rangle c(\Phi^*) c(e_k) - \sum_{k=1}^n \langle e_k, \mathbf{d}x^j \rangle c(e_k) c(\Phi) \right) \\
&\quad + \left[ \sigma_{S(TM) \otimes F}^k + \frac{1}{2} \left( \sum_{l=1}^n \langle e_l, \mathbf{d}x^k \rangle c(\Phi^*) c(e_l) - \sum_{l=1}^n \langle e_l, \mathbf{d}x^k \rangle c(e_l) c(\Phi) \right) \right] \Gamma^k. \tag{2.23}
\end{aligned}$$

Since  $E$  is globally defined on  $M$ , so we can perform computations of  $E$  in normal coordinates. In terms of normal coordinates about  $x_0$  one has:  $\sigma_{S(TM)}^j(x_0) = 0$ ,  $e_j(c(e_i))(x_0) = 0$ ,  $\Gamma^k(x_0) = 0$ , we conclude that

$$\begin{aligned}
E(x_0) &= -\frac{1}{4} s - \frac{1}{2} \sum_{i \neq j} R^F(e_i, e_j) c(e_i) c(e_j) - \frac{1}{4} \sum_i \left[ c(\Phi^*) c(e_i) - c(e_i) c(\Phi) \right]^2 + c(\Phi^*) c(\Phi) \\
&\quad - \frac{1}{2} \sum_j \left[ e_j(c(\Phi^*)) c(e_j) + c(e_j) e_j(c(\Phi)) \right] - \frac{1}{2} \sum_j \left[ \sigma_j^F, c(\Phi^*) \right] c(e_j) - \frac{1}{2} \sum_j c(e_j) \left[ \sigma_j^F, c(\Phi) \right] \\
&= -\frac{1}{4} s - \frac{1}{2} \sum_{i \neq j} R^F(e_i, e_j) c(e_i) c(e_j) - \frac{1}{4} \sum_i \left[ c(\Phi^*) c(e_i) - c(e_i) c(\Phi) \right]^2 + c(\Phi^*) c(\Phi) \\
&\quad - \frac{1}{2} \sum_j \left( \nabla_{e_j}^F c(\Phi^*) \right) c(e_j) - \frac{1}{2} \sum_j c(e_j) \nabla_{e_j}^F c(\Phi). \tag{2.24}
\end{aligned}$$

which, together with (2.13), yields Theorem 2.2.

From Theorem 1 in [6] and Theorem 1 in [7], for  $M$  a compact  $n$  dimensional ( $n \geq 4$ , even) Riemannian manifold and  $\tilde{D}_F^* \tilde{D}_F$  a generalized Laplacian acting on sections of vector bundle on  $M$ , the following relation holds:

$$\text{Wres}(\tilde{D}_F^* \tilde{D}_F)^{\left(\frac{-n+2}{2}\right)} = \frac{(2\pi)^{\frac{n}{2}}}{\left(\frac{n}{2} - 2\right)!} \int_M \text{Tr} \left( \frac{s}{6} + E \right) \text{dvol}_M \tag{2.25}$$

where Wres denotes the noncommutative residue.

By (2.24) and  $Tr(e_i e_j) = 0$  ( $i \neq j$ ), we find for the trace

$$\begin{aligned} \text{Tr}(E) &= \text{Tr} \left[ -\frac{1}{4}s + c(\Phi^*)c(\Phi) - \frac{1}{4} \sum_i [c(\Phi^*)c(e_i) - c(e_i)c(\Phi)]^2 \right. \\ &\quad \left. - \frac{1}{2} \sum_j \nabla_{e_j}^F(c(\Phi^*))c(e_j) - \frac{1}{2} \sum_j c(e_j)\nabla_{e_j}^F(c(\Phi)) \right]. \end{aligned} \quad (2.26)$$

Substituting (2.26) into (2.25), we obtain

**Theorem 2.4.** *For even  $n$ -dimensional compact spin manifolds without boundary, the following equality holds:*

$$\begin{aligned} Wres(\tilde{D}_F^* \tilde{D}_F)^{(-\frac{n+2}{2})} &= \frac{(2\pi)^{\frac{n}{2}}}{(\frac{n}{2}-2)!} \int_M \text{Tr} \left[ -\frac{s}{12} + c(\Phi^*)c(\Phi) - \frac{1}{4} \sum_i [c(\Phi^*)c(e_i) - c(e_i)c(\Phi)]^2 \right. \\ &\quad \left. - \frac{1}{2} \sum_j \nabla_{e_j}^F(c(\Phi^*))c(e_j) - \frac{1}{2} \sum_j c(e_j)\nabla_{e_j}^F(c(\Phi)) \right] dvol_M. \end{aligned} \quad (2.27)$$

where  $s$  is the scalar curvature.

### 3. A Kastler-Kalau-Walze type theorem for 4-dimensional manifolds with boundary associated with twisted Dirac Operators

In this section, we shall prove a Kastler-Kalau-Walze type formula for 4-dimensional compact manifolds with boundary. Some basic facts and formulae about Boutet de Monvel's calculus are recalled as follows.

Let

$$F : L^2(\mathbf{R}_t) \rightarrow L^2(\mathbf{R}_v); F(u)(v) = \int e^{-ivt} u(t) dt$$

denote the Fourier transformation and  $\varphi(\overline{\mathbf{R}^+}) = r^+ \varphi(\mathbf{R})$  (similarly define  $\varphi(\overline{\mathbf{R}^-})$ ), where  $\varphi(\mathbf{R})$  denotes the Schwartz space and

$$r^+ : C^\infty(\mathbf{R}) \rightarrow C^\infty(\overline{\mathbf{R}^+}); f \rightarrow f|_{\overline{\mathbf{R}^+}}; \overline{\mathbf{R}^+} = \{x \geq 0; x \in \mathbf{R}\}. \quad (3.1)$$

We define  $H^+ = F(\varphi(\overline{\mathbf{R}^+}))$ ;  $H_0^- = F(\varphi(\overline{\mathbf{R}^-}))$  which are orthogonal to each other. We have the following property:  $h \in H^+$  ( $H_0^-$ ) iff  $h \in C^\infty(\mathbf{R})$  which has an analytic extension to the lower (upper) complex half-plane  $\{\text{Im} \xi < 0\}$  ( $\{\text{Im} \xi > 0\}$ ) such that for all nonnegative integer  $l$ ,

$$\frac{d^l h}{d\xi^l}(\xi) \sim \sum_{k=1}^{\infty} \frac{d^l}{d\xi^l} \left( \frac{c_k}{\xi^k} \right) \quad (3.2)$$

as  $|\xi| \rightarrow +\infty, \text{Im} \xi \leq 0$  ( $\text{Im} \xi \geq 0$ ).

Let  $H'$  be the space of all polynomials and  $H^- = H_0^- \oplus H'$ ;  $H = H^+ \oplus H^-$ . Denote by  $\pi^+$  ( $\pi^-$ ) respectively the projection on  $H^+$  ( $H^-$ ). For calculations, we take  $H = \tilde{H} = \{\text{rational functions having no poles on the real axis}\}$  ( $\tilde{H}$  is a dense set in the topology of  $H$ ). Then on  $\tilde{H}$ ,

$$\pi^+ h(\xi_0) = \frac{1}{2\pi i} \lim_{u \rightarrow 0^-} \int_{\Gamma^+} \frac{h(\xi)}{\xi_0 + iu - \xi} d\xi, \quad (3.3)$$

where  $\Gamma^+$  is a Jordan close curve included  $\text{Im} \xi > 0$  surrounding all the singularities of  $h$  in the upper half-plane and  $\xi_0 \in \mathbf{R}$ . Similarly, define  $\pi^-$  on  $\tilde{H}$ ,

$$\pi^- h = \frac{1}{2\pi} \int_{\Gamma^+} h(\xi) d\xi. \quad (3.4)$$

So,  $\pi'(H^-) = 0$ . For  $h \in H \cap L^1(R)$ ,  $\pi'h = \frac{1}{2\pi} \int_R h(v) dv$  and for  $h \in H^+ \cap L^1(R)$ ,  $\pi'h = 0$ . Denote by  $\mathcal{B}$  Boutet de Monvel's algebra (for details, see [19]).

In the following, we will compute  $\widetilde{Wres}[\pi^+(\tilde{D}_F^*)^{-1} \circ \pi^+ \tilde{D}_F^{-1}]$ .

Let  $M$  be a compact manifold with boundary  $\partial M$ . We assume that the metric  $g^M$  on  $M$  has the following form near the boundary

$$g^M = \frac{1}{h(x_n)} g^{\partial M} + dx_n^2, \quad (3.5)$$

where  $g^{\partial M}$  is the metric on  $\partial M$ . Let  $U \subset M$  be a collar neighborhood of  $\partial M$  which is diffeomorphic  $\partial M \times [0, 1)$ . By the definition of  $h(x_n) \in C^\infty([0, 1))$  and  $h(x_n) > 0$ , there exists  $\tilde{h} \in C^\infty((-\varepsilon, 1))$  such that  $\tilde{h}|_{[0, 1)} = h$  and  $\tilde{h} > 0$  for some sufficiently small  $\varepsilon > 0$ . Then there exists a metric  $\hat{g}$  on  $\hat{M} = M \cup_{\partial M} \partial M \times (-\varepsilon, 0]$  which has the form on  $U \cup_{\partial M} \partial M \times (-\varepsilon, 0]$

$$\hat{g} = \frac{1}{\tilde{h}(x_n)} g^{\partial M} + dx_n^2, \quad (3.6)$$

such that  $\hat{g}|_M = g$ . We fix a metric  $\hat{g}$  on the  $\hat{M}$  such that  $\hat{g}|_M = g$ . Note  $\tilde{D}_F$  is the twisted Dirac operator on the spinor bundle  $S(TM) \otimes F$  corresponding to the connection  $\tilde{\nabla}$ .

Now we recall the main theorem in [9].

**Theorem 3.1. (Fedosov-Golse-Leichtnam-Schrohe)** *Let  $X$  and  $\partial X$  be connected,  $\dim X = n \geq 3$ ,  $A = \begin{pmatrix} \pi^+ P + G & K \\ T & S \end{pmatrix} \in \mathcal{B}$ , and denote by  $p$ ,  $b$  and  $s$  the local symbols of  $P$ ,  $G$  and  $S$  respectively. Define:*

$$\begin{aligned} \widetilde{Wres}(A) &= \int_X \int_{\mathbf{S}} \text{tr}_E [p_{-n}(x, \xi)] \sigma(\xi) dx \\ &\quad + 2\pi \int_{\partial X} \int_{\mathbf{S}'} \{ \text{tr}_E [(\text{tr} b_{-n})(x', \xi')] + \text{tr}_F [s_{1-n}(x', \xi')] \} \sigma(\xi') dx', \end{aligned} \quad (3.7)$$

Then a)  $\widetilde{Wres}([A, B]) = 0$ , for any  $A, B \in \mathcal{B}$ ; b) It is a unique continuous trace on  $\mathcal{B}/\mathcal{B}^{-\infty}$ .

Denote by  $\sigma_l(A)$  the  $l$ -order symbol of an operator  $A$ . An application of (2.1.4) in [19] shows that

$$\widetilde{Wres}[\pi^+(\tilde{D}_F^*)^{-1} \circ \pi^+ \tilde{D}_F^{-1}] = \int_M \int_{|\xi|=1} \text{trace}_{S(TM) \otimes F} [\sigma_{-n}((\tilde{D}_F^*)^{-1} \circ \tilde{D}_F^{-1})] \sigma(\xi) dx + \int_{\partial M} \Psi, \quad (3.8)$$

where

$$\begin{aligned} \Psi &= \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{j,k=0}^{\infty} \sum \frac{(-i)^{|\alpha|+j+k+\ell}}{\alpha!(j+k+1)!} \text{trace}_{S(TM) \otimes F} \left[ \partial_{x_n}^j \partial_{\xi'}^\alpha \partial_{\xi_n}^k \sigma_r^+((\tilde{D}_F^*)^{-1})(x', 0, \xi', \xi_n) \right. \\ &\quad \left. \times \partial_{x_n}^\alpha \partial_{\xi_n}^{j+1} \partial_{x_n}^k \sigma_l(\tilde{D}_F^{-1})(x', 0, \xi', \xi_n) \right] d\xi_n \sigma(\xi') dx', \end{aligned} \quad (3.9)$$

and the sum is taken over  $r - k + |\alpha| + \ell - j - 1 = -n, r \leq -1, \ell \leq -1$ .

Locally we can use Theorem 2.4 to compute the interior term of (3.8), then

$$\begin{aligned} &\int_M \int_{|\xi|=1} \text{trace}_{S(TM) \otimes F} [\sigma_{-4}((\tilde{D}_F^*)^{-1} \circ \tilde{D}_F^{-1})] \sigma(\xi) dx \\ &= 4\pi^2 \int_M \text{Tr} \left[ -\frac{s}{12} + c(\Phi^*)c(\Phi) - \frac{1}{4} \sum_i [c(\Phi^*)c(e_i) - c(e_i)c(\Phi)]^2 - \frac{1}{2} \sum_j \nabla_{e_j}^F (c(\Phi^*))c(e_j) \right. \\ &\quad \left. - \frac{1}{2} \sum_j c(e_j) \nabla_{e_j}^F (c(\Phi)) \right] d\text{vol}_M. \end{aligned} \quad (3.10)$$

So we only need to compute  $\int_{\partial M} \Psi$ . Let us now turn to compute the symbol expansion of  $\tilde{D}_F^{-1}$ . Recall the definition of the twisted Dirac operator  $\tilde{D}_F$  in Definition 2.1. Let  $\nabla^{TM}$  denote the Levi-civita connection about  $g^M$ . In the local coordinates  $\{x_i; 1 \leq i \leq n\}$  and the fixed orthonormal frame  $\{\tilde{e}_1, \dots, \tilde{e}_n\}$ , the connection matrix  $(\omega_{s,t})$  is defined by

$$\nabla^{TM}(\tilde{e}_1, \dots, \tilde{e}_n) = (\tilde{e}_1, \dots, \tilde{e}_n)(\omega_{s,t}). \quad (3.11)$$

Let  $c(\tilde{e}_i)$  denote the Clifford action. Let  $g^{ij} = g(dx_i, dx_j)$  and

$$\nabla_{\partial_i}^{TM} \partial_j = \sum_k \Gamma_{ij}^k \partial_k; \quad \Gamma^k = g^{ij} \Gamma_{ij}^k. \quad (3.12)$$

Let the cotangent vector  $\xi = \sum \xi_j dx_j$  and  $\xi^j = g^{ij} \xi_i$ . By Lemma 1 in [11] and Lemma 2.1 in [10], for any fixed point  $x_0 \in \partial M$ , we can choose the normal coordinates  $U$  of  $x_0$  in  $\partial M$  (not in  $M$ ). By the composition formula and (2.2.11) in [10], we obtain

**Lemma 3.2.** *Let  $\tilde{D}_F^*, \tilde{D}_F$  be the twisted Dirac operators on  $\Gamma(S(TM) \otimes F)$ . Then*

$$\sigma_{-1}((\tilde{D}_F^*)^{-1}) = \sigma_{-1}(\tilde{D}_F^{-1}) = \frac{\sqrt{-1}c(\xi)}{|\xi|^2}; \quad (3.13)$$

$$\sigma_{-2}((\tilde{D}_F^*)^{-1}) = \frac{c(\xi)\sigma_0(\tilde{D}_F^*)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} \sum_j c(dx_j) \left[ \partial_{x_j}[c(\xi)]|\xi|^2 - c(\xi)\partial_{x_j}(|\xi|^2) \right]; \quad (3.14)$$

$$\sigma_{-2}(\tilde{D}_F^{-1}) = \frac{c(\xi)\sigma_0(\tilde{D}_F)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} \sum_j c(dx_j) \left[ \partial_{x_j}[c(\xi)]|\xi|^2 - c(\xi)\partial_{x_j}(|\xi|^2) \right], \quad (3.15)$$

where

$$\sigma_0(\tilde{D}_F^*) = \sigma_0(D) + \sum_{j=1}^n c(e_j)(\sigma_j^F - \Phi^*(e_j)); \quad (3.16)$$

$$\sigma_0(\tilde{D}_F) = \sigma_0(D) + \sum_{j=1}^n c(e_j)(\sigma_j^F + \Phi(e_j)). \quad (3.17)$$

Let us now turn to compute  $\Psi$  (see formula (3.9) for definition of  $\Psi$ ). Since the sum is taken over  $-r - \ell + k + j + |\alpha| = 3$ ,  $r, \ell \leq -1$ , then we have the boundary term of (3.8) is the sum of the following five terms.

**case a) I)**  $r = -1, \ell = -1, k = j = 0, |\alpha| = 1$

From (3.9) we have

$$\text{case a) I)} = - \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{trace} \left[ \partial_{\xi'}^\alpha \pi_{\xi_n}^+ \sigma_{-1}((\tilde{D}_F^*)^{-1}) \times \partial_{x'}^\alpha \partial_{\xi_n} \sigma_{-1}(\tilde{D}_F^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx'. \quad (3.18)$$

By Lemma 2.2 in [10], for  $i < n$ , then

$$\partial_{x_i} \sigma_{-1}(\tilde{D}_F^{-1})(x_0) = \partial_{x_i} \left( \frac{\sqrt{-1}c(\xi)}{|\xi|^2} \right) (x_0) = \frac{\sqrt{-1}\partial_{x_i}[c(\xi)](x_0)}{|\xi|^2} - \frac{\sqrt{-1}c(\xi)\partial_{x_i}(|\xi|^2)(x_0)}{|\xi|^4} = 0, \quad (3.19)$$

so **case a) I)** vanishes.

**case a) II)**  $r = -1, \ell = -1, k = |\alpha| = 0, j = 1$

From (3.9) we have

$$\text{case a) II)} = - \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[ \partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}((\tilde{D}_F^*)^{-1}) \times \partial_{\xi_n}^2 \sigma_{-1}(\tilde{D}_F^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx', \quad (3.20)$$



An application of Lemma 2.1 and Lemma 2.2 in [10] shows that

$$\partial_{\xi_n}^2 \sigma_{-1}(\tilde{D}_F^{-1}) = \sqrt{-1} \left( -\frac{6\xi_n c(\mathbf{d}x_n) + 2c(\xi')}{|\xi|^4} + \frac{8\xi_n^2 c(\xi)}{|\xi|^6} \right); \quad (3.21)$$

and

$$\partial_{x_n} \sigma_{-1}((\tilde{D}_F^*)^{-1})(x_0) = \frac{\sqrt{-1} \partial_{x_n} c(\xi')(x_0)}{|\xi|^2} - \frac{\sqrt{-1} c(\xi) |\xi'|^2 h'(0)}{|\xi|^4}. \quad (3.22)$$

By (2.1) in [10] and the Cauchy integral formula, we obtain

$$\begin{aligned} \pi_{\xi_n}^+ \left[ \frac{c(\xi)}{|\xi|^4} \right] (x_0)|_{|\xi'|=1} &= \pi_{\xi_n}^+ \left[ \frac{c(\xi') + \xi_n c(\mathbf{d}x_n)}{(1 + \xi_n^2)^2} \right] \\ &= \frac{1}{2\pi i} \lim_{u \rightarrow 0^-} \int_{\Gamma^+} \frac{\frac{c(\xi') + \eta_n c(\mathbf{d}x_n)}{(\eta_n + i)^2 (\xi_n + iu - \eta_n)}}{(\eta_n - i)^2} d\eta_n \\ &= \left[ \frac{c(\xi') + \eta_n c(\mathbf{d}x_n)}{(\eta_n + i)^2 (\xi_n - \eta_n)} \right]^{(1)} \Big|_{\eta_n = i} \\ &= -\frac{ic(\xi')}{4(\xi_n - i)} - \frac{c(\xi') + ic(\mathbf{d}x_n)}{4(\xi_n - i)^2}. \end{aligned} \quad (3.23)$$

Similarly,

$$\pi_{\xi_n}^+ \left[ \frac{\sqrt{-1} \partial_{x_n} c(\xi')}{|\xi|^2} \right] (x_0)|_{|\xi'|=1} = \frac{\partial_{x_n} [c(\xi')](x_0)}{2(\xi_n - i)}. \quad (3.24)$$

Combining (3.23) and (3.24), we have

$$\pi_{\xi_n}^+ \partial_{x_n} \sigma_{-1}(D_F^{-1})(x_0)|_{|\xi'|=1} = \frac{\partial_{x_n} [c(\xi')](x_0)}{2(\xi_n - i)} + \sqrt{-1} h'(0) \left[ \frac{ic(\xi')}{4(\xi_n - i)} + \frac{c(\xi') + ic(\mathbf{d}x_n)}{4(\xi_n - i)^2} \right]. \quad (3.25)$$

By the relation of the Clifford action and  $\text{tr} AB = \text{tr} BA$ , we have the equalities:

$$\begin{aligned} \text{tr}[c(\xi') c(\mathbf{d}x_n)] &= 0; \quad \text{tr}[c(\mathbf{d}x_n)^2] = -4\dim F; \quad \text{tr}[c(\xi')^2](x_0)|_{|\xi'|=1} = -4\dim F; \\ \text{tr}[\partial_{x_n} c(\xi') c(\mathbf{d}x_n)] &= 0; \quad \text{tr}[\partial_{x_n} c(\xi') c(\xi')](x_0)|_{|\xi'|=1} = -2h'(0)\dim F. \end{aligned} \quad (3.26)$$

From equation (3.21), (3.25) and (3.26), one sees that

$$\begin{aligned} \text{case a) II)} &= - \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{ih'(0)(\xi_n - i)^2}{(\xi_n - i)^4 (\xi_n + i)^3} d\xi_n \sigma(\xi') d\mathbf{x}' \\ &= -ih'(0) \Omega_3 \int_{\Gamma^+} \frac{1}{(\xi_n - i)^2 (\xi_n + i)^3} d\xi_n d\mathbf{x}' \\ &= -ih'(0) \Omega_3 2\pi i \left[ \frac{1}{(\xi_n + i)^3} \right]^{(1)} \Big|_{\xi_n = i} d\mathbf{x}' \\ &= -\frac{3}{8} \pi h'(0) \dim F \Omega_3 d\mathbf{x}'. \end{aligned} \quad (3.27)$$

**case a) III)**  $r = -1, l = -1, j = |\alpha| = 0, k = 1$

From (3.9) we have

$$\text{case a) III)} = -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[ \partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}((\tilde{D}_F^*)^{-1}) \times \partial_{\xi_n} \partial_{x_n} \sigma_{-1}(\tilde{D}_F^{-1}) \right] (x_0) d\xi_n \sigma(\xi') d\mathbf{x}', \quad (3.28)$$

Then an application of Lemma 2.2 in [10] shows

$$\partial_{\xi_n} \partial_{x_n} q_{-1}(x_0)|_{|\xi'|=1} = -\sqrt{-1} h'(0) \left[ \frac{c(\mathbf{d}x_n)}{|\xi|^4} - 4\xi_n \frac{c(\xi') + \xi_n c(\mathbf{d}x_n)}{|\xi|^6} \right] - \frac{2\xi_n \sqrt{-1} \partial_{x_n} c(\xi')(x_0)}{|\xi|^4}, \quad (3.29)$$

and

$$\partial_{\xi_n} \pi_{\xi_n}^+ q_{-1}(x_0)|_{|\xi'|=1} = -\frac{c(\xi') + ic(\mathbf{d}x_n)}{2(\xi_n - i)^2}. \quad (3.30)$$

Similar to **case a) II**), we obtain

$$\mathbf{case\ a)\ III} = \frac{3}{8} \pi h'(0) \dim F \Omega_3 \mathbf{d}x'. \quad (3.31)$$

**case b)**  $r = -2, l = -1, k = j = |\alpha| = 0$

From (3.9) we have

$$\begin{aligned} \text{case b)} &= -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[ \pi_{\xi_n}^+ \sigma_{-2}((\tilde{D}_F^*)^{-1}) \times \partial_{\xi_n} \sigma_{-1}(\tilde{D}_F^{-1}) \right] (x_0) \mathbf{d}\xi_n \sigma(\xi') \mathbf{d}x' \\ &= -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[ \pi_{\xi_n}^+ \sigma_{-2}(D^{-1}) \times \partial_{\xi_n} \sigma_{-1}(\tilde{D}_F^{-1}) \right] (x_0) \mathbf{d}\xi_n \sigma(\xi') \mathbf{d}x' \\ &\quad -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[ \pi_{\xi_n}^+ \left( \frac{c(\xi) \alpha c(\xi)}{|\xi|^4} \right) \times \partial_{\xi_n} \sigma_{-1}(\tilde{D}_F^{-1}) \right] (x_0) \mathbf{d}\xi_n \sigma(\xi') \mathbf{d}x'. \end{aligned} \quad (3.32)$$

where  $\alpha = \sum_{j=1}^4 c(e_j)(\sigma_j^F - \Phi^*(e_j))$ .

From (3.45) in [13], we have

$$-i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} [\pi_{\xi_n}^+ \sigma_{-2}(D^{-1}) \times \partial_{\xi_n} \sigma_{-1}(\tilde{D}_F^{-1})] (x_0) \mathbf{d}\xi_n \sigma(\xi') \mathbf{d}x' = \frac{9}{8} h'(0) \pi \dim F \Omega_3 \mathbf{d}x'. \quad (3.33)$$

On the other hand,

$$\pi_{\xi_n}^+ \left[ \frac{c(\xi) \alpha c(\xi)}{|\xi|^4} \right] = \frac{(-i\xi_n - 2)c(\xi') \alpha c(\xi') - i \left[ c(\mathbf{d}x_n) \alpha c(\xi') + c(\xi') \alpha c(\mathbf{d}x_n) \right] - i\xi_n c(\mathbf{d}x_n) \alpha c(\mathbf{d}x_n)}{4(\xi_n - i)^2}. \quad (3.34)$$

By the relation of the Clifford action and  $\mathbf{tr} AB = \mathbf{tr} BA$ , then we have the equalities

$$\mathbf{tr} \left[ c(\mathbf{d}x_n) \sum_{j=1}^4 c(e_j)(\sigma_j^F - \Phi^*(e_j)) \right] = \mathbf{tr} \left[ -\mathbf{id} \otimes (\sigma_n^F - \Phi^*(e_n)) \right], \quad (3.35)$$

$$\mathbf{tr} \left[ c(\xi') \sum_{j=1}^4 c(e_j)(\sigma_j^F - \Phi^*(e_j)) \right] = \mathbf{tr} \left[ -\sum_{j=1}^3 \xi_j (\sigma_j^F - \Phi^*(e_j)) \right]. \quad (3.36)$$

We note that  $i < n$ ,  $\int_{|\xi'|=1} \xi_i \sigma(\xi') = 0$ , so (3.36) has no contribution for computing case b). Hence

$$\begin{aligned} &-i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[ \pi_{\xi_n}^+ \left[ \frac{c(\xi) \alpha c(\xi)}{|\xi|^4} \right] \times \partial_{\xi_n} \sigma_{-1}(\tilde{D}_F^{-1}) \right] (x_0) \mathbf{d}\xi_n \sigma(\xi') \mathbf{d}x' \\ &= -\frac{1}{4} \pi \Omega_3 \mathbf{Tr} [\mathbf{id} \otimes (\sigma_n^F - \Phi^*(e_j))] \mathbf{d}x'. \end{aligned} \quad (3.37)$$

Combining (3.34) and (3.37), we have

$$\text{case b)} = \left[ \frac{9}{8} h'(0) \dim F - \frac{1}{4} \mathbf{Tr} (\mathbf{id} \otimes (\sigma_n^F - \Phi^*(e_j))) \right] \pi \Omega_3 \mathbf{d}x'. \quad (3.38)$$

**case c)**  $r = -1, l = -2, k = j = |\alpha| = 0$

From (3.9) we have

$$\begin{aligned}
\text{case c)} &= -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[ \pi_{\xi_n}^+ \sigma_{-1}((\tilde{D}_F^*)^{-1}) \times \partial_{\xi_n} \sigma_{-2}(\tilde{D}_F^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx' \\
&= -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[ \pi_{\xi_n}^+ \sigma_{-1}((\tilde{D}_F^*)^{-1}) \times \partial_{\xi_n} \sigma_{-2}(D^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx' \\
&\quad -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[ \pi_{\xi_n}^+ \sigma_{-1}((\tilde{D}_F^*)^{-1}) \times \partial_{\xi_n} \left( \frac{c(\xi)\beta c(\xi)}{|\xi|^4} \right) \right] (x_0) d\xi_n \sigma(\xi') dx', \quad (3.39)
\end{aligned}$$

where  $\beta = \sum_{j=1}^n c(e_j)(\sigma_j^F + \Phi(e_j))$ .

From (3.50) in [13], we have

$$-i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[ \pi_{\xi_n}^+ \sigma_{-1}((\tilde{D}_F^*)^{-1}) \times \partial_{\xi_n} \sigma_{-2}(\tilde{D}_F^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx' = -\frac{9}{8} h'(0) \pi \dim F \Omega_3 dx'. \quad (3.40)$$

On the other hand,

$$\pi_{\xi_n}^+ \sigma_{-1}((\tilde{D}_F^*)^{-1}) = \frac{c(\xi') + ic(\mathbf{d}x_n)}{2(\xi_n - i)}. \quad (3.41)$$

By the relation of the Clifford action and  $\text{tr}AB = \text{tr}BA$ , then we have the equalities

$$\text{tr} \left[ c(\mathbf{d}x_n) \sum_{j=1}^4 c(e_j)(\sigma_j^F + \Phi(e_j)) \right] = \text{tr} \left[ -\text{id} \otimes (\sigma_n^F + \Phi(e_n)) \right], \quad (3.42)$$

$$\text{tr} \left[ c(\xi') \sum_{j=1}^4 c(e_j)(\sigma_j^F + \Phi(e_j)) \right] = \text{tr} \left[ -\sum_{j=1}^3 \xi_j (\sigma_j^F + \Phi(e_j)) \right]. \quad (3.43)$$

From (3.41) and (3.42) we obtain

$$\begin{aligned}
&-i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[ \pi_{\xi_n}^+ \sigma_{-1}((\tilde{D}_F^*)^{-1}) \times \partial_{\xi_n} \left( \frac{c(\xi)\beta c(\xi)}{|\xi|^4} \right) \right] (x_0) d\xi_n \sigma(\xi') dx' \\
&= \frac{1}{4} \pi \Omega_3 \text{Tr}[\text{id} \otimes (\sigma_n^F + \Phi(e_n))] dx'. \quad (3.44)
\end{aligned}$$

Combining (3.40) and (3.44), we have

$$\text{case c)} = \left[ -\frac{9}{8} h'(0) \dim F + \frac{1}{4} \text{Tr}[\text{id} \otimes (\sigma_n^F + \Phi(e_n))] \right] \pi \Omega_3 dx'. \quad (3.45)$$

We note that  $\dim S(TM) = 4$ , now  $\Psi$  is the sum of the **case (a, b, c)**, so

$$\sum \text{case a, b, c} = \text{Tr}_F(\Phi^*(e_n) + \Phi(e_n)) \pi \Omega_3 dx'. \quad (3.46)$$

Hence we conclude that

**Theorem 3.3.** *Let  $M$  be a 4-dimensional compact manifolds with the boundary  $\partial M$ . Then*

$$\begin{aligned}
\widetilde{Wres}[\pi^+(\tilde{D}_F^*)^{-1} \circ \pi^+ \tilde{D}_F^{-1}] &= 4\pi^2 \int_M \text{Tr} \left[ -\frac{s}{12} + c(\Phi^*)c(\Phi) - \frac{1}{4} \sum_i [c(\Phi^*)c(e_i) - c(e_i)c(\Phi)]^2 \right. \\
&\quad \left. - \frac{1}{2} \sum_j \nabla_{e_j}^F (c(\Phi^*))c(e_j) - \frac{1}{2} \sum_j c(e_j) \nabla_{e_j}^F (c(\Phi)) \right] d\text{vol}_M \\
&\quad + \int_{\partial M} \text{Tr}_F(\Phi^*(e_n) + \Phi(e_n)) \pi \Omega_3 dx', \quad (3.47)
\end{aligned}$$

where  $s$  is the scalar curvature.

#### 4. A Lichnerowicz formula for twisted signature operators

Let us recall the definition of twisted signature operators. We consider a  $n$ -dimensional oriented Riemannian manifold  $(M, g^M)$ . Let  $F$  be a real vector bundle over  $M$ . Let  $g^F$  be an Euclidean metric on  $F$ . Let

$$\wedge^*(T^*M) = \bigoplus_{i=0}^n \wedge^i(T^*M) \quad (4.1)$$

be the real exterior algebra bundle of  $T^*M$ . Let

$$\Omega^*(M, F) = \bigoplus_{i=0}^n \Omega^i(M, F) = \bigoplus_{i=0}^n C^\infty(M, \wedge^i(T^*M) \otimes F) \quad (4.2)$$

be the set of smooth sections of  $\wedge^*(T^*M) \otimes F$ . Let  $*$  be the Hodge star operator of  $g^{TM}$ . It extends on  $\wedge^*(T^*M) \otimes F$  by acting on  $F$  as identity. Then  $\Omega^*(M, F)$  inherits the following standardly induced inner product

$$\langle \alpha, \beta \rangle = \int_M \langle \alpha \wedge * \beta \rangle_F, \quad \alpha, \beta \in \Omega^*(M, F). \quad (4.3)$$

Let  $\widehat{\nabla}^F$  be the non-Euclidean connection on  $F$ . Let  $d^F$  be the obvious extension of  $\nabla^F$  on  $\Omega^*(M, F)$ . Let  $\delta^F = d^{F*}$  be the formal adjoint operator of  $d^F$  with respect to the inner product. Let  $\hat{D}^F$  be the differential operator acting on  $\Omega^*(M, F)$  defined by

$$\hat{D}^F = d^F + \delta^F. \quad (4.4)$$

Let

$$\omega(F, g^F) = \widehat{\nabla}^{F,*} - \widehat{\nabla}^F, \quad \nabla^{F,e} = \nabla^F + \frac{1}{2}\omega(F, g^F). \quad (4.5)$$

Then  $\nabla^{F,e}$  is an Euclidean connection on  $(F, g^F)$ .

Let  $\nabla^{\wedge^*(T^*M)}$  be the Euclidean connection on  $\wedge^*(T^*M)$  induced canonically by the Levi-Civita connection  $\nabla^{TM}$  of  $g^{TM}$ . Let  $\nabla^e$  be the Euclidean connection on  $\wedge^*(T^*M) \otimes F$  obtained from the tensor product of  $\nabla^{\wedge^*(T^*M)}$  and  $\nabla^{F,e}$ . Let  $\{e_1, \dots, e_n\}$  be an oriented (local) orthonormal basis of  $TM$ . The following result was proved by Proposition in [14].

**Proposition 4.1.** [14] *The following identity holds*

$$d^F + \delta^F = \sum_{i=1}^n c(e_i) \nabla_{e_i}^e - \frac{1}{2} \sum_{i=1}^n \hat{c}(e_i) \omega(F, g^F)(e_i). \quad (4.6)$$

Let

$$D_F^e = \sum_{j=1}^n c(e_j) \nabla_{e_j}^e, \quad (4.7)$$

then the twisted signature operators  $\hat{D}_F, \hat{D}_F^*$  as follows.

**Definition 4.2.** *For sections  $\psi \otimes \chi \in \wedge^*(T^*M) \otimes F$ ,*

$$\hat{D}_F(\psi \otimes \chi) = D_F^e(\psi \otimes \chi) - \frac{1}{2} \sum_{i=1}^n \hat{c}(e_i) \omega(F, g^F)(e_i)(\psi \otimes \chi), \quad (4.8)$$

$$\hat{D}_F^*(\psi \otimes \chi) = D_F^{*,e}(\psi \otimes \chi) - \frac{1}{2} \sum_{i=1}^n \hat{c}(e_i) \omega^*(F, g^F)(e_i)(\psi \otimes \chi). \quad (4.9)$$

Here  $\omega^*(F, g^F)(e_i)$  denotes the adjoint of  $\omega(F, g^F)(e_i)$ .

We first establish the main theorem in this section. Let  $\hat{c}(\omega) = \sum_i c(e_i)\omega(F, g^F)(e_i)$  and  $\hat{c}(\omega^*) = \sum_i c(e_i)\omega^*(F, g^F)(e_i)$ , then

**Theorem 4.3.** *The following identity holds*

$$\begin{aligned} \hat{D}_F^* \hat{D}_F &= -\left[g^{ij}(\nabla'_{\partial_i} \nabla'_{\partial_j} - \nabla'_{\nabla_{\partial_i}^L \partial_j})\right] + \frac{1}{4}s + \frac{1}{2} \sum_{i \neq j} R^{F,e}(e_i, e_j)c(e_i)c(e_j) + \frac{1}{4}\hat{c}(\omega^*)\hat{c}(\omega) \\ &\quad + \frac{1}{4} \sum_i \nabla_{e_i}^F \hat{c}(\omega^*)c(e_i) - \frac{1}{4} \sum_i c(e_i) \nabla_{e_i}^F \hat{c}(\omega) + \frac{1}{16} \sum_i \left[\hat{c}(\omega^*)c(e_i) + c(e_i)\hat{c}(\omega)\right]^2, \end{aligned} \quad (4.10)$$

where  $s$  is the scalar curvature,  $R^{F,e}$  denotes the curvature-tensor on  $F$ .

Now we shall prove Theorem 4.3. Similar to (2.20), we have

$$\begin{aligned} \hat{D}_F^* \hat{D}_F &= -g^{ij}\partial_i \partial_j - 2\sigma_{\wedge^*(T^*M) \otimes F}^j \partial_j + \Gamma^k \partial_k - \frac{1}{2} \sum_j \left[\hat{c}(\omega^*)c(e_j) + c(e_j) \otimes \hat{c}(\omega)\right] e_j \\ &\quad - g^{ij} \left[\partial_i (\sigma_{\wedge^*(T^*M) \otimes F}^{j,e}) + \sigma_{\wedge^*(T^*M) \otimes F}^i \sigma_{\wedge^*(T^*M) \otimes F, e}^{j,e} - \Gamma_{ij}^k \sigma_{\wedge^*(T^*M) \otimes F}^k\right] \\ &\quad - \frac{1}{2} \sum_j \hat{c}(\omega^*)c(e_j) \sigma_j^{\wedge^*(T^*M) \otimes F, e} - \frac{1}{2} \sum_j c(e_j) \otimes e_j (\hat{c}(\omega)) \\ &\quad - \frac{1}{2} \sum_j c(e_j) \sigma_j^{\wedge^*(T^*M) \otimes F, e} \otimes \hat{c}(\omega) + \frac{1}{4} \hat{c}(\omega^*) \hat{c}(\omega) \\ &\quad + \frac{1}{4}s + \frac{1}{2} \sum_{i \neq j} R^{F,e}(e_i, e_j)c(e_i)c(e_j). \end{aligned} \quad (4.11)$$

In terms of local coordinates  $\{\partial_i\}$  inducing the coordinate transformation  $e_j = \sum_{k=1}^n \langle e_j, \mathbf{d}x^k \rangle \partial_k$ , then

$$\omega_j = \sigma_{\wedge^*(T^*M)}^j + \sigma_F^{j,e} + \frac{1}{4} \left( \sum_{k=1}^n \langle e_k, \mathbf{d}x^j \rangle \hat{c}(\omega^*(F, g^F))c(e_k) - \sum_{k=1}^n \langle e_k, \mathbf{d}x^j \rangle c(e_k) \hat{c}(\omega(F, g^F)) \right) + \frac{1}{2} \Gamma^j. \quad (4.12)$$

For a smooth vector field  $X \in \Gamma(M, TM)$ , then

$$\nabla'_X = \nabla_X^{\wedge^*(T^*M) \otimes F, e} + \frac{1}{4} [\hat{c}(\omega^*(F, g^F))c(X) - c(X)\hat{c}(\omega(F, g^F))]. \quad (4.13)$$

Since  $E$  is globally defined on  $M$ , so we can perform computations of  $E$  in normal coordinates. In terms of normal coordinates about  $x_0$  one has:  $\sigma_{\wedge^*(T^*M)}^j(x_0) = 0$ ,  $e_j(c(e_i))(x_0) = 0$ ,  $\Gamma^k(x_0) = 0$ . From (2.15) and (4.12), we obtain

$$\begin{aligned} E(x_0) &= -\frac{1}{4}s - \frac{1}{2} \sum_{i \neq j} R^{F,e}(e_i, e_j)c(e_i)c(e_j) - \frac{1}{16} \sum_i \left[\hat{c}(\omega^*)c(e_i) - c(e_i)\hat{c}(\omega)\right]^2 - \frac{1}{4}\hat{c}(\omega^*)\hat{c}(\omega) \\ &\quad - \frac{1}{4} \sum_j \left[e_j(\hat{c}(\omega^*))c(e_j) - c(e_j)e_j(\hat{c}(\omega))\right] - \frac{1}{4} \sum_j \left[\sigma_j^F, \hat{c}(\omega^*)\right]c(e_j) + \frac{1}{4} \sum_j c(e_j) \left[\sigma_j^F, \hat{c}(\omega)\right] \\ &= -\frac{1}{4}s - \frac{1}{2} \sum_{i \neq j} R^{F,e}(e_i, e_j)c(e_i)c(e_j) - \frac{1}{16} \sum_i \left[\hat{c}(\omega^*)c(e_i) - c(e_i)\hat{c}(\omega)\right]^2 - \frac{1}{4}\hat{c}(\omega^*)\hat{c}(\omega) \\ &\quad - \frac{1}{4} \sum_j \nabla_{e_j}^F (\hat{c}(\omega^*))c(e_j) + \frac{1}{4} \sum_j c(e_j) \nabla_{e_j}^F (\hat{c}(\omega)). \end{aligned} \quad (4.14)$$

which, together with (2.13), yields Theorem 4.3.

From (4.14) we have

$$\begin{aligned}
\text{Tr}(E) &= \text{Tr} \left\{ -\frac{1}{4}s - \frac{1}{2} \sum_{i \neq j} R^{F,e}(e_i, e_j) c(e_i) c(e_j) - \frac{1}{16} \sum_i \left[ \hat{c}(\omega^*) c(e_i) - c(e_i) \hat{c}(\omega) \right]^2 \right. \\
&\quad \left. - \frac{1}{4} \hat{c}(\omega^*) \hat{c}(\omega) - \frac{1}{4} \sum_j \nabla_{e_j}^F (\hat{c}(\omega^*)) c(e_j) + \frac{1}{4} \sum_j c(e_j) \nabla_{e_j}^F (\hat{c}(\omega)) \right\} \\
&= \text{Tr} \left[ -\frac{1}{4}s + \frac{n}{16} [\hat{c}(\omega^*) - \hat{c}(\omega)]^2 - \frac{1}{4} \hat{c}(\omega^*) \hat{c}(\omega) - \frac{1}{4} \sum_j \nabla_{e_j}^F (\hat{c}(\omega^*)) c(e_j) \right. \\
&\quad \left. + \frac{1}{4} \sum_j c(e_j) \nabla_{e_j}^F (\hat{c}(\omega)) \right]. \tag{4.15}
\end{aligned}$$

Hence we conclude that

**Theorem 4.4.** *For even  $n$ -dimensional oriented compact Riemannian manifolds without boundary, the following equality holds:*

$$\begin{aligned}
\text{Wres}(\hat{D}_F^* \hat{D}_F)^{\left(\frac{-n+2}{2}\right)} &= \frac{(2\pi)^{\frac{n}{2}}}{\left(\frac{n}{2}-2\right)!} \int_M \text{Tr} \left[ -\frac{s}{12} + \frac{n}{16} [\hat{c}(\omega^*) - \hat{c}(\omega)]^2 - \frac{1}{4} \hat{c}(\omega^*) \hat{c}(\omega) \right. \\
&\quad \left. - \frac{1}{4} \sum_j \nabla_{e_j}^F (\hat{c}(\omega^*)) c(e_j) + \frac{1}{4} \sum_j c(e_j) \nabla_{e_j}^F (\hat{c}(\omega)) \right] \text{dvol}_M. \tag{4.16}
\end{aligned}$$

## 5. A Kastler-Kalau-Walze theorem for 4-dimensional Riemannian manifolds with boundary associated to twisted Signature Operators

In this section, we shall prove a Kastler-Kalau-Walze type formula for  $\hat{D}_F^* \hat{D}_F$ . An application of (2.1.4) in [19] shows that

$$\widetilde{\text{Wres}}[\pi^+(\hat{D}_F^*)^{-1} \circ \pi^+(\hat{D}_F)^{-1}] = \int_M \int_{|\xi|=1} \text{trace}_{\wedge^*(T^*M) \otimes F} [\sigma_{-n}((\hat{D}_F^*)^{-1} \circ (\hat{D}_F)^{-1})] \sigma(\xi) \text{d}x + \int_{\partial M} \tilde{\Psi}, \tag{5.1}$$

where

$$\begin{aligned}
\tilde{\Psi} &= \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{j,k=0}^{\infty} \sum \frac{(-i)^{|\alpha|+j+k+\ell}}{\alpha!(j+k+1)!} \text{trace}_{\wedge^*(T^*M) \otimes F} \left[ \partial_{x_n}^j \partial_{\xi'}^\alpha \partial_{\xi_n}^k \sigma_r^+((\hat{D}_F^*)^{-1})(x', 0, \xi', \xi_n) \right. \\
&\quad \left. \times \partial_{x_n}^\alpha \partial_{\xi_n}^{j+1} \partial_{x_n}^k \sigma_l((\hat{D}_F)^{-1})(x', 0, \xi', \xi_n) \right] \text{d}\xi_n \sigma(\xi') \text{d}x', \tag{5.2}
\end{aligned}$$

and the sum is taken over  $r - k + |\alpha| + \ell - j - 1 = -n, r \leq -1, \ell \leq -1$ .

Locally we can use Theorem 4.4 to compute the interior term of (5.1), then

$$\begin{aligned}
&\int_M \int_{|\xi|=1} \text{trace}_{\wedge^*(T^*M) \otimes F} [\sigma_{-4}((\hat{D}_F^*)^{-1} \circ (\hat{D}_F)^{-1})] \sigma(\xi) \text{d}x \\
&= 4\pi^2 \int_M \text{Tr} \left[ -\frac{s}{12} + \frac{1}{4} [\hat{c}(\omega^*) - \hat{c}(\omega)]^2 - \frac{1}{4} \hat{c}(\omega^*) \hat{c}(\omega) - \frac{1}{4} \sum_j \nabla_{e_j}^F (\hat{c}(\omega^*)) c(e_j) \right. \\
&\quad \left. + \frac{1}{4} \sum_j c(e_j) \nabla_{e_j}^F (\hat{c}(\omega)) \right] \text{dvol}_M. \tag{5.3}
\end{aligned}$$

So we only need to compute  $\int_{\partial M} \tilde{\Psi}$ . In the local coordinates  $\{x_i; 1 \leq i \leq n\}$  and the fixed orthonormal frame  $\{\tilde{e}_1, \dots, \tilde{e}_n\}$ , the connection matrix  $(\omega_{s,t})$  is defined by

$$\tilde{\nabla}(\tilde{e}_1, \dots, \tilde{e}_n) = (\tilde{e}_1, \dots, \tilde{e}_n)(\omega_{s,t}). \tag{5.4}$$

Let  $M$  be a 4-dimensional compact oriented Riemannian manifold with boundary  $\partial M$  and the metric of (6.1).  $\hat{D}_F = d^F + \delta^F : C^\infty(M, \wedge^*(T^*M) \otimes F) \rightarrow C^\infty(M, \wedge^*(T^*M) \otimes F)$  is the twisted signature operator. Take the coordinates and the orthonormal frame as in Section 3. Let  $\epsilon(\widetilde{e_j^*})$ ,  $\iota(\widetilde{e_j^*})$  be the exterior and interior multiplications respectively. Write

$$c(\widetilde{e_j}) = \epsilon(\widetilde{e_j^*}) - \iota(\widetilde{e_j^*}); \quad \hat{c}(\widetilde{e_j}) = \epsilon(\widetilde{e_j^*}) + \iota(\widetilde{e_j^*}). \quad (5.5)$$

We'll compute  $\text{tr}_{\wedge^*(T^*M) \otimes F}$  in the frame  $\{e_{i_1}^* \wedge \cdots \wedge e_{i_k}^* \mid 1 \leq i_1 < \cdots < i_k \leq 4\}$ . By (3.2) in [10], we have

$$\begin{aligned} \hat{D}_F &= d^F + \delta^F = \sum_{i=1}^n c(e_i) \nabla_{e_i}^e - \frac{1}{2} \sum_{i=1}^n \hat{c}(e_i) \omega(F, g^F)(e_i) \\ &= \sum_{i=1}^n c(e_i) \left( \nabla_{e_i}^{\wedge^*(T^*M)} \otimes id_F + id_{\wedge^*(T^*M)} \otimes \nabla_{e_i}^{F,e} \right) - \frac{1}{2} \sum_{i=1}^n \hat{c}(e_i) \omega(F, g^F)(e_i) \\ &= \sum_{i=1}^n c(\widetilde{e_i}) \left[ \widetilde{e_i} + \frac{1}{4} \sum_{s,t} \omega_{s,t}(\widetilde{e_i}) [\hat{c}(\widetilde{e_s}) \hat{c}(\widetilde{e_t}) - c(\widetilde{e_s}) c(\widetilde{e_t})] \otimes id_F \right. \\ &\quad \left. + id_{\wedge^*(T^*M)} \otimes \sigma_i^{F,e} \right] - \frac{1}{2} \sum_{i=1}^n \hat{c}(e_i) \omega(F, g^F)(e_i), \end{aligned} \quad (5.6)$$

$$\begin{aligned} \hat{D}_F^* &= \sum_{i=1}^n c(\widetilde{e_i}) \left[ \widetilde{e_i} + \frac{1}{4} \sum_{s,t} \omega_{s,t}(\widetilde{e_i}) [\hat{c}(\widetilde{e_s}) \hat{c}(\widetilde{e_t}) - c(\widetilde{e_s}) c(\widetilde{e_t})] \otimes id_F \right. \\ &\quad \left. + id_{\wedge^*(T^*M)} \otimes \sigma_i^{F,e} \right] - \frac{1}{2} \sum_{i=1}^n \hat{c}(e_i) \omega^*(F, g^F)(e_i). \end{aligned} \quad (5.7)$$

Then we obtain

$$\sigma_1(\hat{D}_F) = \sigma_1(\hat{D}_F^*) = \sqrt{-1}c(\xi); \quad (5.8)$$

$$\begin{aligned} \sigma_0(\hat{D}_F) &= \sum_{i=1}^n c(\widetilde{e_i}) \left[ \frac{1}{4} \sum_{s,t} \omega_{s,t}(\widetilde{e_i}) [\hat{c}(\widetilde{e_s}) \hat{c}(\widetilde{e_t}) - c(\widetilde{e_s}) c(\widetilde{e_t})] \otimes id_F + id_{\wedge^*(T^*M)} \otimes \sigma_i^{F,e} \right] \\ &\quad - \frac{1}{2} \sum_{i=1}^n \hat{c}(e_i) \omega(F, g^F)(e_i); \end{aligned} \quad (5.9)$$

$$\begin{aligned} \sigma_0(\hat{D}_F^*) &= \sum_{i=1}^n c(\widetilde{e_i}) \left[ \frac{1}{4} \sum_{s,t} \omega_{s,t}(\widetilde{e_i}) [\hat{c}(\widetilde{e_s}) \hat{c}(\widetilde{e_t}) - c(\widetilde{e_s}) c(\widetilde{e_t})] \otimes id_F + id_{\wedge^*(T^*M)} \otimes \sigma_i^{F,e} \right] \\ &\quad - \frac{1}{2} \sum_{i=1}^n \hat{c}(e_i) \omega^*(F, g^F)(e_i). \end{aligned} \quad (5.10)$$

By the composition formula of pseudodifferential operators in Section 2.2.1 of [10], we have

**Lemma 5.1.** *The symbol of the twisted signature operators  $\hat{D}_F^*, \hat{D}_F$  as follows:*

$$\sigma_{-1}((\hat{D}_F)^{-1}) = \sigma_{-1}((\hat{D}_F^*)^{-1}) = \frac{\sqrt{-1}c(\xi)}{|\xi|^2}; \quad (5.11)$$

$$\sigma_{-2}((\hat{D}_F)^{-1}) = \frac{c(\xi)\sigma_0(\hat{D}_F)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} \sum_j c(dx_j) \left[ \partial_{x_j}(c(\xi))|\xi|^2 - c(\xi)\partial_{x_j}(|\xi|^2) \right]; \quad (5.12)$$

$$\sigma_{-2}((\hat{D}_F^*)^{-1}) = \frac{c(\xi)\sigma_0(\hat{D}_F^*)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} \sum_j c(dx_j) \left[ \partial_{x_j}(c(\xi))|\xi|^2 - c(\xi)\partial_{x_j}(|\xi|^2) \right]. \quad (5.13)$$

Since  $\tilde{\Psi}$  is a global form on  $\partial M$ , so for any fixed point  $x_0 \in \partial M$ , we can choose the normal coordinates  $U$  of  $x_0$  in  $\partial M$  (not in  $M$ ) and compute  $\tilde{\Psi}(x_0)$  in the coordinates  $\tilde{U} = U \times [0, 1)$  and the metric  $\frac{1}{h(x_n)}g^{\partial M} + dx_n^2$ . The dual metric of  $g^{\partial M}$  on  $\tilde{U}$  is  $\frac{1}{h(x_n)}g^{\partial M} + dx_n^2$ . Write  $g_{ij}^M = g^M(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ ;  $g_M^{ij} = g^M(dx_i, dx_j)$ , then

$$[g_{i,j}^M] = \begin{bmatrix} \frac{1}{h(x_n)}[g_{i,j}^{\partial M}] & 0 \\ 0 & 1 \end{bmatrix}; \quad [g_M^{i,j}] = \begin{bmatrix} h(x_n)[g_{\partial M}^{i,j}] & 0 \\ 0 & 1 \end{bmatrix}, \quad (5.14)$$

and

$$\partial_{x_n} g_{ij}^{\partial M}(x_0) = 0, \quad 1 \leq i, j \leq n-1; \quad g_{i,j}^M(x_0) = \delta_{ij}. \quad (5.15)$$

Let  $\{e_1, \dots, e_{n-1}\}$  be an orthonormal frame field in  $U$  about  $g^{\partial M}$  which is parallel along geodesics and  $e_i = \frac{\partial}{\partial x_i}(x_0)$ , then  $\{\tilde{e}_1 = \sqrt{h(x_n)}e_1, \dots, \tilde{e}_{n-1} = \sqrt{h(x_n)}e_{n-1}, \tilde{e}_n = dx_n\}$  is the orthonormal frame field in  $\tilde{U}$  about  $g^M$ . Locally  $\wedge^*(T^*M)|_{\tilde{U}} \cong \tilde{U} \times \wedge_C^*(\frac{n}{2})$ . Let  $\{f_1, \dots, f_n\}$  be the orthonormal basis of  $\wedge_C^*(\frac{n}{2})$ . Take a spin frame field  $\sigma : \tilde{U} \rightarrow Spin(M)$  such that  $\pi\sigma = \{\tilde{e}_1, \dots, \tilde{e}_n\}$  where  $\pi : Spin(M) \rightarrow O(M)$  is a double covering, then  $\{[\sigma, f_i], 1 \leq i \leq 4\}$  is an orthonormal frame of  $\wedge^*(T^*M)|_{\tilde{U}}$ . In the following, since the global form  $\tilde{\Psi}$  is independent of the choice of the local frame, so we can compute  $\text{tr}_{\wedge^*(T^*M)}$  in the frame  $\{[\sigma, f_i], 1 \leq i \leq 4\}$ . Let  $\{E_1, \dots, E_n\}$  be the canonical basis of  $R^n$  and  $c(E_i) \in cl_C(n) \cong \text{Hom}(\wedge_C^*(\frac{n}{2}), \wedge_C^*(\frac{n}{2}))$  be the Clifford action. By [10], then

$$c(\tilde{e}_i) = [(\sigma, c(E_i))]; \quad c(\tilde{e}_i)[(\sigma, f_i)] = [\sigma, (c(E_i)f_i)]; \quad \frac{\partial}{\partial x_i} = [(\sigma, \frac{\partial}{\partial x_i})], \quad (5.16)$$

then we have  $\frac{\partial}{\partial x_i}c(\tilde{e}_i) = 0$  in the above frame. By Lemma 2.2 in [10], we have

**Lemma 5.2.** *With the metric  $g^M$  on  $M$  near the boundary*

$$\partial_{x_j}(|\xi|_{g^M}^2)(x_0) = \begin{cases} 0, & \text{if } j < n; \\ h'(0)|\xi'|_{g^{\partial M}}^2, & \text{if } j = n. \end{cases} \quad (5.17)$$

$$\partial_{x_j}[c(\xi)](x_0) = \begin{cases} 0, & \text{if } j < n; \\ \partial_{x_n}(c(\xi'))(x_0), & \text{if } j = n, \end{cases} \quad (5.18)$$

where  $\xi = \xi' + \xi_n dx_n$

Then an application of Lemma 2.3 in [10] shows

**Lemma 5.3.** *The symbol of the twisted signature operators  $\hat{D}_F^*, \hat{D}_F$*

$$\begin{aligned} \sigma_0(\hat{D}_F^*) &= -\frac{3}{4}h'(0)c(dx_n) + \frac{1}{4}h'(0) \sum_{i=1}^{n-1} c(\tilde{e}_i)\hat{c}(\tilde{e}_n)\hat{c}(\tilde{e}_i)(x_0) \otimes id_F \\ &\quad + \sum_{i=1}^n c(\tilde{e}_i)\sigma_i^{F,e} - \frac{1}{2} \sum_{i=1}^n \hat{c}(e_i)\omega^*(F, g^F)(e_i); \end{aligned} \quad (5.19)$$

$$\begin{aligned} \sigma_0(\hat{D}_F) &= -\frac{3}{4}h'(0)c(dx_n) + \frac{1}{4}h'(0) \sum_{i=1}^{n-1} c(\tilde{e}_i)\hat{c}(\tilde{e}_n)\hat{c}(\tilde{e}_i)(x_0) \otimes id_F \\ &\quad + \sum_{i=1}^n c(\tilde{e}_i)\sigma_i^{F,e} - \frac{1}{2} \sum_{i=1}^n \hat{c}(e_i)\omega(F, g^F)(e_i) \end{aligned} \quad (5.20)$$

Now we can compute  $\tilde{\Psi}$  (see formula (5.2) for definition of  $\tilde{\Psi}$ ), since the sum is taken over  $-r - \ell + k + j + |\alpha| = 3$ ,  $r, \ell \leq -1$ , then we have the following five cases:

**Case a(I):**  $r = -1, \ell = -1, k = j = 0, |\alpha| = 1$



From (5.2), we have

$$\text{Case a(I)} = - \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{trace}[\partial_{\xi'}^{\alpha} \pi_{\xi_n}^{+} \sigma_{-1}((\hat{D}_F^{*})^{-1}) \partial_{x'}^{\alpha} \partial_{\xi_n} \sigma_{-1}((\hat{D}_F)^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \quad (5.21)$$

By Lemma 5.2, for  $j < n$

$$\partial_{x_j} \sigma_{-1}((\hat{D}_F)^{-1})(x_0) = \partial_{x_j} \left( \frac{\sqrt{-1}c(\xi)}{|\xi|^2} \right) (x_0) = \frac{\sqrt{-1}\partial_{x_i}[c(\xi)](x_0)}{|\xi|^2} - \frac{\sqrt{-1}c(\xi)\partial_{x_i}(|\xi|^2)(x_0)}{|\xi|^4} = 0, \quad (5.22)$$

so Case a(I) vanishes.

**Case a(II):**  $r = -1, \ell = -1, k = |\alpha| = 0, j = 1$

From (5.2), we have

$$\text{Case a(II)} = -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{x_n} \pi_{\xi_n}^{+} \sigma_{-1}((\hat{D}_F^{*})^{-1}) \partial_{\xi_n}^2 \sigma_{-1}((\hat{D}_F)^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \quad (5.23)$$

Similar to (2.2.18) in [10], we have

$$\partial_{x_n} \pi_{\xi_n}^{+} \sigma_{-1}((\hat{D}_F^{*})^{-1})(x_0)|_{|\xi'|=1} = \frac{\partial_{x_n}[c(\xi')](x_0)}{2(\xi_n - i)} + \sqrt{-1}h'(0) \left[ \frac{ic(\xi')}{4(\xi_n - i)} + \frac{c(\xi') + ic(dx_n)}{4(\xi_n - i)^2} \right]; \quad (5.24)$$

and

$$\partial_{\xi_n}^2 \sigma_{-1}((\hat{D}_F)^{-1}) = \sqrt{-1} \left( -\frac{6\xi_n c(dx_n) + 2c(\xi')}{|\xi|^4} + \frac{8\xi_n^2 c(\xi)}{|\xi|^6} \right). \quad (5.25)$$

Let  $\dim F = l$ , by the relation of the Clifford action and  $\text{tr} AB = \text{tr} BA$ , then

$$\begin{aligned} \text{tr}[c(\xi')c(dx_n)] &= 0; \quad \text{tr}[c(dx_n)^2] = -16l; \quad \text{tr}[c(\xi')^2](x_0)|_{|\xi'|=1} = -16l; \\ \text{tr}[\partial_{x_n}[c(\xi')]c(dx_n)] &= 0; \quad \text{tr}[\partial_{x_n}c(\xi') \times c(\xi')](x_0)|_{|\xi'|=1} = -8lh'(0). \end{aligned} \quad (5.26)$$

For more trace expansions, we can see [20].

Then

$$\text{trace} \left[ \partial_{x_n} \pi_{\xi_n}^{+} \sigma_{-1}((\hat{D}_F^{*})^{-1}) \partial_{\xi_n}^2 \sigma_{-1}((\hat{D}_F)^{-1}) \right] (x_0) = \frac{8lih'(0)}{(\xi_n - i)^2(\xi_n + i)^3}. \quad (5.27)$$

Therefore

$$\begin{aligned} \text{case a(II)} &= - \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{4lih'(0)(\xi_n - i)^2}{(\xi_n - i)^4(\xi_n + i)^3} d\xi_n \sigma(\xi') dx' \\ &= -4lih'(0)\Omega_3 \int_{\Gamma^+} \frac{1}{(\xi_n - i)^2(\xi_n + i)^3} d\xi_n dx' \\ &= -4lih'(0)\Omega_3 2\pi i \left[ \frac{1}{(\xi_n + i)^3} \right]^{(1)} \Big|_{\xi_n=i} dx' \\ &= -\frac{3l}{2}\pi h'(0)\Omega_3 dx', \end{aligned} \quad (5.28)$$

where  $\Omega_3$  is the canonical volume of  $S^3$ .

**Case a(III):**  $r = -1, \ell = -1, j = |\alpha| = 0, k = 1$

From (5.2), we have

$$\text{Case a(III)} = -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{\xi_n} \pi_{\xi_n}^{+} \sigma_{-1}((\hat{D}_F^{*})^{-1}) \partial_{\xi_n} \partial_{x_n} \sigma_{-1}((\hat{D}_F)^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \quad (5.29)$$

Similar to (2.2.27) in [10], we have

$$\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}((\hat{D}_F^*)^{-1})(x_0)|_{|\xi'|=1} = -\frac{c(\xi') + ic(dx_n)}{2(\xi_n - i)^2}, \quad (5.30)$$

and

$$\partial_{\xi_n} \partial_{x_n} \sigma_{-1}((\hat{D}_F)^{-1})(x_0)|_{|\xi'|=1} = -\sqrt{-1}h'(0) \left[ \frac{c(dx_n)}{|\xi|^4} - 4\xi_n \frac{c(\xi') + \xi_n c(dx_n)}{|\xi|^6} \right] - \frac{2\sqrt{-1}\xi_n \partial_{x_n} c(\xi')(x_0)}{|\xi|^4}. \quad (5.31)$$

Combining (5.30) and (5.31), we obtain

$$\text{trace} \left[ \partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}((\hat{D}_F^*)^{-1}) \partial_{\xi_n} \partial_{x_n} \sigma_{-1}((\hat{D}_F)^{-1}) \right] (x_0) = \frac{8lh'(0)(i - 2\xi_n - i\xi_n^2)}{(\xi_n - i)^4(\xi_n + i)^3}. \quad (5.32)$$

Then

$$\text{Case a(III)} = \frac{3l}{2} \pi h'(0) \Omega_3 dx', \quad (5.33)$$

where  $\Omega_3$  is the canonical volume of  $S^3$ . Thus the sum of Case a(II) and Case a(III) is zero.

**Case b:**  $r = -2$ ,  $\ell = -1$ ,  $k = j = |\alpha| = 0$

By (5.2), we get

$$\text{Case b} = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} [\pi_{\xi_n}^+ \sigma_{-2}((\hat{D}_F^*)^{-1}) \partial_{\xi_n} \sigma_{-1}((\hat{D}_F)^{-1})] (x_0) d\xi_n \sigma(\xi') dx'. \quad (5.34)$$

Then an application of Lemma 5.2 shows

$$\begin{aligned} \sigma_{-2}((\hat{D}_F^*)^{-1})(x_0) &= \frac{c(\xi) \sigma_0(D^F)(x_0) c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} \sum_j c(dx_j) \left[ \partial_{x_j}(c(\xi)) |\xi|^2 - c(\xi) \partial_{x_j}(|\xi|^2) \right] (x_0) \\ &= \frac{c(\xi) \sigma_0(D^F)(x_0) c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} c(dx_n) \left[ \partial_{x_n}(c(\xi'))(x_0) - c(\xi) h'(0) |\xi'|_{g^{\partial M}}^2 \right]. \end{aligned} \quad (5.35)$$

Hence,

$$\pi_{\xi_n}^+ \sigma_{-2}((\hat{D}_F^*)^{-1})(x_0) := A_1 + A_2, \quad (5.36)$$

where

$$\begin{aligned} A_1 &= -\frac{h'(0)}{2} \left[ \frac{c(dx_n)}{4i(\xi_n - i)} + \frac{c(dx_n) - ic(\xi')}{8(\xi_n - i)^2} + \frac{3\xi_n - 7i}{8(\xi_n - i)^3} [ic(\xi') - c(dx_n)] \right] \\ &\quad + \frac{-1}{4(\xi_n - i)^2} \left[ (2 + i\xi_n) c(\xi') \alpha_0 c(\xi') + i\xi_n c(dx_n) \alpha_0 c(dx_n) + (2 + i\xi_n) c(\xi') c(dx_n) \partial_{x_n} c(\xi') \right. \\ &\quad \left. + ic(dx_n) \alpha_0 c(\xi') + ic(\xi') \alpha_0 c(dx_n) - i \partial_{x_n} c(\xi') \right]; \end{aligned} \quad (5.37)$$

$$\begin{aligned} A_2 &= \frac{-1}{4(\xi_n - i)^2} \left[ (2 + i\xi_n) c(\xi') \beta_0 c(\xi') + i\xi_n c(dx_n) \beta_0 c(dx_n) + ic(dx_n) \beta_0 c(\xi') \right. \\ &\quad \left. + ic(\xi') \beta_0 c(dx_n) \right] \end{aligned} \quad (5.38)$$

and

$$\alpha_0 = -\frac{3}{4} h'(0) c(dx_n) + \frac{1}{4} h'(0) \sum_{i=1}^{n-1} c(\tilde{e}_i) \hat{c}(\tilde{e}_n) \hat{c}(\tilde{e}_i)(x_0) \otimes id_F := -\frac{3}{4} h'(0) c(dx_n) + p_0 \otimes id_F, \quad (5.39)$$

$$\beta_0 = \sum_{i=1}^n c(\tilde{e}_i) \sigma_i^{F,e} - \frac{1}{2} \sum_{i=1}^n \hat{c}(e_i) \omega^*(F, g^F)(e_i). \quad (5.40)$$

On the other hand,

$$\partial_{\xi_n} \sigma_{-1}((\hat{D}_F)^{-1}) = \frac{-2i\xi_n c(\xi')}{(1 + \xi_n^2)^2} + \frac{i(1 - \xi_n^2) c(dx_n)}{(1 + \xi_n^2)^2}. \quad (5.41)$$

For the signature operator case,

$$\text{tr}[c(\xi') \alpha_0 c(\xi') c(dx_n)](x_0) = \text{tr}[\alpha_0 c(\xi') c(dx_n) c(\xi')](x_0) = |\xi'|^2 \text{tr}[\alpha_0 c(dx_n)], \quad (5.42)$$

and

$$\begin{aligned} c(dx_n) p_0(x_0) &= -\frac{1}{4} h'(0) \sum_{i=1}^{n-1} c(\tilde{e}_i) \hat{c}(\tilde{e}_i) c(\tilde{e}_n) \hat{c}(\tilde{e}_n) \\ &= -\frac{1}{4} h'(0) \sum_{i=1}^{n-1} [\epsilon(\tilde{e}_i^*) \iota(\tilde{e}_i^*) - \iota(\tilde{e}_i^*) \epsilon(\tilde{e}_i^*)] [\epsilon(\tilde{e}_n^*) \iota(\tilde{e}_n^*) - \iota(\tilde{e}_n^*) \epsilon(\tilde{e}_n^*)]. \end{aligned} \quad (5.43)$$

By Section 3 in [10], then

$$\begin{aligned} &\text{tr}_{\wedge^m(T^*M)} \{ [\epsilon(e_i^*) \iota(e_i^*) - \iota(e_i^*) \epsilon(e_i^*)] [\epsilon(e_n^*) \iota(e_n^*) - \iota(e_n^*) \epsilon(e_n^*)] \} \\ &= a_{n,m} \langle e_i^*, e_n^* \rangle^2 + b_{n,m} |e_i^*|^2 |e_n^*|^2 = b_{n,m}, \end{aligned} \quad (5.44)$$

where  $b_{4,m} = \binom{2}{m-2} + \binom{2}{m} - 2 \binom{2}{m-1}$ . Then

$$\text{tr}_{\wedge^*(T^*M)} \{ [\epsilon(\tilde{e}_i^*) \iota(\tilde{e}_i^*) - \iota(\tilde{e}_i^*) \epsilon(\tilde{e}_i^*)] [\epsilon(\tilde{e}_n^*) \iota(\tilde{e}_n^*) - \iota(\tilde{e}_n^*) \epsilon(\tilde{e}_n^*)] \} = \sum_{m=0}^4 b_{4,m} = 0. \quad (5.45)$$

Hence in this case,

$$\text{tr}_{\wedge^*(T^*M)} [c(dx_n) p_0(x_0)] = 0. \quad (5.46)$$

We note that  $\int_{|\xi'|=1} \xi_1 \cdots \xi_{2q+1} \sigma(\xi') = 0$ , then  $\text{tr}_{\wedge^*(T^*M)} [c(\xi') p_0(x_0)]$  has no contribution for computing Case b.

From (2.2.39), (2.2.41) and (2.2.42) in [10], we have

$$-i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[A_1 \times \partial_{\xi_n} \sigma_{-1}((\hat{D}_F)^{-1})](x_0) d\xi_n \sigma(\xi') dx' = \frac{9l}{2} \pi h'(0) \Omega_3 dx'. \quad (5.47)$$

Combining (5.38) and (5.41), we obtain

$$\text{trace}[A_2 \times \partial_{\xi_n} \sigma_{-1}((\hat{D}_F)^{-1})](x_0) = \frac{-1 - 2i\xi_n + \xi_n^2}{2(\xi_n + i)^2(\xi_n - i)^4} \text{tr}[\beta_0 c(\xi')] + \frac{-i + 2\xi_n - i\xi_n^2}{2(\xi_n + i)^2(\xi_n - i)^4} \text{tr}[\beta_0 c(dx_n)]. \quad (5.48)$$

By the relation of the Clifford action and  $\text{tr}AB = \text{tr}BA$ , then we have the equalities

$$\text{tr}[c(\tilde{e}_i) c(dx_n)] = 0, i < n; \quad \text{tr}[c(\tilde{e}_i) c(dx_n)] = -16l, i = n; \quad \text{tr}[\hat{c}(\tilde{e}_i) c(\xi')] = \text{tr}[\hat{c}(\tilde{e}_i) c(dx_n)] = 0. \quad (5.49)$$

Then  $\text{tr}[\beta_0 c(\xi')]$  has no contribution for computing Case b.

From (5.48) and (5.49), we have

$$\begin{aligned} &-i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[A_2 \times \partial_{\xi_n} \sigma_{-1}((\hat{D}_F)^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &= 2i\Omega_3 \int_{\Gamma^+} \frac{-i + 2\xi_n - i\xi_n^2}{2(\xi_n + i)^2(\xi_n - i)^4} \text{tr}[id_{\wedge^*(T^*M)} \otimes \sigma_n^{F,e}] d\xi_n dx' \\ &= 2i \frac{2\pi i}{3!} \Omega_3 \left[ \frac{-i + 2\xi_n - i\xi_n^2}{2(\xi_n + i)^2} \right]^{(1)} \Big|_{\xi_n=i} \text{tr}[id_{\wedge^*(T^*M)} \otimes \sigma_n^{F,e}] dx' \\ &= -4\pi \text{tr}_F[\sigma_n^{F,e}] \Omega_3 dx'. \end{aligned} \quad (5.50)$$

Combining (5.47) and (5.50), we have

$$\text{case } b = \left[ \frac{9}{2} l h'(0) - 4 \text{tr}_F[\sigma_n^{F,e}] \right] \pi \Omega_3 \mathbf{d}x'. \quad (5.51)$$

**Case c:**  $r = -1$ ,  $\ell = -2$ ,  $k = j = |\alpha| = 0$

From (5.2), we have

$$\text{case } c = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ \sigma_{-1}((\hat{D}_F^*)^{-1}) \partial_{\xi_n} \sigma_{-2}((\hat{D}_F)^{-1})](x_0) \mathbf{d}\xi_n \sigma(\xi') \mathbf{d}x'. \quad (5.52)$$

From (5.13) we obtain

$$\pi_{\xi_n}^+ \sigma_{-1}((\hat{D}_F^*)^{-1}) = \frac{c(\xi') + i c(\mathbf{d}x_n)}{2(\xi_n - i)}. \quad (5.53)$$

Hence,

$$\partial_{\xi_n} \sigma_{-2}((\hat{D}_F)^{-1})(x_0) := B_1 + B_2, \quad (5.54)$$

where

$$\begin{aligned} B_1 = & \frac{1}{(1 + \xi_n^2)^3} \left[ (2\xi_n - 2\xi_n^3) c(\mathbf{d}x_n) \alpha_0 c(\mathbf{d}x_n) + (1 - 3\xi_n^2) c(\mathbf{d}x_n) \alpha_0 c(\xi') \right. \\ & + (1 - 3\xi_n^2) c(\xi') \alpha_0 c(\mathbf{d}x_n) - 4\xi_n c(\xi') \alpha_0 c(\xi') + (3\xi_n^2 - 1) \partial_{x_n} c(\xi') - 4\xi_n c(\xi') c(\mathbf{d}x_n) \partial_{x_n} c(\xi') \\ & \left. + 2h'(0) c(\xi') + 2h'(0) \xi_n c(\mathbf{d}x_n) \right] + 6\xi_n h'(0) \frac{c(\xi) c(\mathbf{d}x_n) c(\xi)}{(1 + \xi_n^2)^4}; \end{aligned} \quad (5.55)$$

$$\begin{aligned} B_2 = & \frac{1}{(1 + \xi_n^2)^3} \left[ (2\xi_n - 2\xi_n^3) c(\mathbf{d}x_n) \beta_0 c(\mathbf{d}x_n) + (1 - 3\xi_n^2) c(\mathbf{d}x_n) \beta_0 c(\xi') \right. \\ & \left. + (1 - 3\xi_n^2) c(\xi') \beta_0 c(\mathbf{d}x_n) - 4\xi_n c(\xi') \beta_0 c(\xi') \right]. \end{aligned} \quad (5.56)$$

Similar to **Case b**, we have

$$-i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ \sigma_{-1}((\hat{D}_F^*)^{-1}) \times B_1](x_0) \mathbf{d}\xi_n \sigma(\xi') \mathbf{d}x' = -\frac{9l}{2} \pi h'(0) \Omega_3 \mathbf{d}x'. \quad (5.57)$$

From (5.53) and (5.56), we obtain

$$\begin{aligned} \text{trace}[\pi_{\xi_n}^+ \sigma_{-1}((\hat{D}_F^*)^{-1}) \times B_2](x_0) = & \frac{-i + 3\xi_n + 3i\xi_n^2 - \xi_n^3}{(\xi_n + i)^3(\xi_n - i)^4} \text{tr}[\beta_0 c(\xi')] \\ & + \frac{-1 - 3i\xi_n + 3\xi_n^2 + i\xi_n^3}{(\xi_n + i)^3(\xi_n - i)^4} \text{tr}[\beta_0 c(\mathbf{d}x_n)]. \end{aligned} \quad (5.58)$$

Hence in this case,

$$\begin{aligned} & -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ \sigma_{-1}((\hat{D}_F^*)^{-1}) \times B_2](x_0) \mathbf{d}\xi_n \sigma(\xi') \mathbf{d}x' \\ = & 2i\Omega_3 \int_{\Gamma^+} \frac{-1 - 3i\xi_n + 3\xi_n^2 + i\xi_n^3}{(\xi_n + i)^3(\xi_n - i)^4} \text{tr}[id_{\wedge^*(T^*M)} \otimes \sigma_n^{F,e}] \mathbf{d}\xi_n \mathbf{d}x' \\ = & 2i \frac{2\pi i}{3!} \Omega_3 \left[ \frac{-1 - 3i\xi_n + 3\xi_n^2 + i\xi_n^3}{(\xi_n + i)^3} \right]^{(1)} \Big|_{\xi_n=i} \text{tr}[id_{\wedge^*(T^*M)} \otimes \sigma_n^{F,e}] \mathbf{d}x' \\ = & 4\pi \text{tr}_F[\sigma_n^{F,e}] \Omega_3 \mathbf{d}x'. \end{aligned} \quad (5.59)$$

Combining (5.57) and (5.59), we obtain

$$\text{case } c = \left[ -\frac{9}{2} l h'(0) + 4 \text{tr}_F[\sigma_n^{F,e}] \right] \pi \Omega_3 \mathbf{d}x'. \quad (5.60)$$

Now  $\widetilde{\Psi}$  is the sum of the **case (a, b, c)**, so

$$\sum \text{case a, b, c} = 0. \quad (5.61)$$

Hence we conclude that

**Theorem 5.4.** *Let  $M$  be a 4-dimensional compact oriented Riemannian manifold and the metric  $g^M$  as above, let  $\hat{D}_F^*, \hat{D}_F$  be the twisted signature operators on  $C^\infty(M, \wedge^*(T^*M) \otimes F)$ , then*

$$\begin{aligned} \widetilde{Wres}[\pi^+(\hat{D}_F^*)^{-1}\pi^+(\hat{D}_F)^{-1}] &= 4\pi^2 \int_M \text{Tr} \left[ -\frac{s}{12} + \frac{1}{4}[\hat{c}(\omega^*) - \hat{c}(\omega)]^2 - \frac{1}{4}\hat{c}(\omega^*)\hat{c}(\omega) \right. \\ &\quad \left. - \frac{1}{4} \sum_j \nabla_{e_j}^F(\hat{c}(\omega^*))c(e_j) + \frac{1}{4} \sum_j c(e_j)\nabla_{e_j}^F(\hat{c}(\omega)) \right] d\text{vol}_M. \end{aligned} \quad (5.62)$$

where  $s$  be the scalar curvature of  $M$ .

Now we give a Kastler-Kalau-Walze theorem for  $\hat{D}_F^2$ . let  $\Delta^e$  be the Bochner laplacian

$$\Delta^e = \sum_{i=1}^n \left[ (\nabla_{e_i}^e)^2 - \nabla_{\nabla_{e_i}^{TM} e_i}^e \right]. \quad (5.63)$$

From Theorem 1.1 in [16], we can state the following Lichnerowicz type formula for  $D_F^2$ .

**Theorem 5.5.** [16] *The following identity holds*

$$\begin{aligned} \hat{D}_F^2 &= -\Delta^e + \frac{s}{4} - \frac{1}{8} \sum_{i=1}^n c(e_i)c(e_j)(\omega(F, g^F))^2(e_i, e_j) \\ &\quad + \frac{1}{8} \sum_{i,j=1}^n \langle R^{TM}(e_i, e_j)e_i, e_j \rangle c(e_i)c(e_j)\hat{c}(e_i)\hat{c}(e_j) \\ &\quad + \frac{1}{4} \sum_{j=1}^n (\omega(F, g^F)(e_j))^2 + \frac{1}{8} \sum_{i,j=1}^n \hat{c}(e_i)\hat{c}(e_j)(\omega(F, g^F))^2(e_i, e_j) \\ &\quad - \frac{1}{4} \sum_{i,j=1}^n c(e_i)\hat{c}(e_j) \left( \nabla_{e_i}^F \omega(F, g^F)(e_j) + \nabla_{e_j}^F \omega(F, g^F)(e_i) \right) \\ &:= -\Delta^e + E. \end{aligned} \quad (5.64)$$

Therefore

$$\text{tr}[E] = \text{tr} \left[ \frac{s}{4} + \frac{1}{2}(\omega(F, g^F)(e_i))^2 \right]. \quad (5.65)$$

Then we obtain

**Theorem 5.6.** *Let  $M$  be a 4-dimensional compact oriented Riemannian manifold without boundary, and  $\hat{D}_F = d^F + \delta^F$  be the twisted signature operator on  $C^\infty(M, \wedge^*(T^*M) \otimes F)$ , then*

$$Wres[\hat{D}_F^{-2}] = 2\pi^2 \int_M \text{tr} \left[ \frac{5}{6}s + (\omega(F, g^F)(e_i))^2 \right] d\text{vol}_M. \quad (5.66)$$

**Theorem 5.7.** *Let  $M$  be a 4-dimensional compact oriented Riemannian manifold with boundary, and  $\hat{D}_F = d^F + \delta^F$  be the twisted signature operator on  $C^\infty(M, \wedge^*(T^*M) \otimes F)$ , then*

$$Wres[(\pi^+(\hat{D}_F)^{-1})^2] = 2\pi^2 \int_M \text{tr} \left[ \frac{5}{6}s + (\omega(F, g^F)(e_i))^2 \right] d\text{vol}_M. \quad (5.67)$$

## Acknowledgements

This work was supported by Fok Ying Tong Education Foundation under Grant No. 121003, NSFC. 11271062 and NCET-13-0721. The authors also thank the referee for his (or her) careful reading and helpful comments.

## References

## References

- [1] V. W. Guillemin.: A new proof of Weyl's formula on the asymptotic distribution of eigenvalues. *Adv. Math.* 55, no. 2, 131-160, (1985).
- [2] M. Wodzicki.: local invariants of spectral asymmetry. *Invent. Math.* 75(1), 143-178, (1995).
- [3] M. Adler.: On a trace functional for formal pseudo-differential operators and the symplectic structure of Korteweg-de Vries type equations, *Invent. Math.* 50, 219-248,(1979).
- [4] A. Connes.: Quantized calculus and applications. XIth International Congress of Mathematical Physics(Paris,1994), Internat Press, Cambridge, MA, 15-36, (1995).
- [5] A. Connes.: The action functional in Noncommutative geometry. *Comm. Math. Phys.* 117, 673-683, (1998).
- [6] D. Kastler.: The Dirac Operator and Gravitation. *Comm. Math. Phys.* 166, 633-643, (1995).
- [7] W. Kalau and M. Walze.: Gravity, Noncommutative geometry and the Wodzicki residue. *J. Geom. Phys.* 16, 327-344, (1995).
- [8] T. Ackermann.: A note on the Wodzicki residue. *J. Geom. Phys.* 20, 404-406, (1996).
- [9] B. V. Fedosov, F. Golse, E. Leichtnam, E. Schrohe.: The noncommutative residue for manifolds with boundary. *J. Funct. Anal.* 142, 1-31, (1996).
- [10] Y. Wang.: Gravity and the Noncommutative Residue for Manifolds with Boundary. *Lett. Math. Phys.* 80, 37-56, (2007).
- [11] Y. Wang.: Lower-Dimensional Volumes and Kastler-kalau-Walze Type Theorem for Manifolds with Boundary . *Commun. Theor. Phys.* Vol 54, 38-42, (2010).
- [12] J. Wang and Y. Wang.: Nonminimal operators and non-commutative residue, *J. Math. Phys.* 53, 072503 (2012).
- [13] J. Wang and Y. Wang.: Noncommutative residue and sub-Dirac operators for foliations, *J. Math. Phys.* 54, 012501 (2013).
- [14] J. M. Bismut and W. Zhang.: An Extension of a theorem by Cheeger and Müller, *Astérisque*, No. 205, paris, (1992).
- [15] X.N. MA and W. ZHANG.:  $\eta$ -Invariant and Flat Vector Bundles. *Chin. Ann. Math.* 27B(1), 67-72, (2006).
- [16] W. Zhang.: Sub-signature operators,  $\eta$  invariants and a Riemann-Roch theorem for flat vector bundles. *Chin. Ann. Math.* 25B: 1, 7-36, (2004).
- [17] P. B. Gilkey, *Invariance Theory, the Heat Equation, and the Atiyah-Singer Index Theorem*, vol. 11 of Mathematics Lecture Series, 1984.
- [18] T. Ackermann and J. Tolksdorf.: A generalized Lichnerowicz formula, the Wodzicki residue and gravity. *J. Geom. Phys.* 19, 143-150,(1996).
- [19] Y. Wang.: Differential forms and the Wodzicki residue for Manifolds with Boundary. *J. Geom. Phys.* 56, 731-753, (2006).
- [20] G. Grubb, E. Schrohe.: Trace expansions and the noncommutative residue for manifolds with boundary. *J. Reine Angew. Math.* 536, 167-207, (2001).